Lie–Poisson simulation of the dynamics of point vortices on a rotating sphere coupled with a background field

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Contents

Motivations

- The point vortex dynamics on a sphere model is central in understanding atmospheric and oceanic dynamics
- The coupled model is necessarily introduced when the sphere is rotating
- A structure preserving numerical scheme (Lie–Poisson integrator) for the coupled model is still missing
- The dynamics and the classification of the relative equilibria configurations of the point vortices together with the background vorticity are still not well known

Euler equations on a rotating sphere

- Lie–Poisson Hamiltonian system on

$$sdiff^*(\mathbb{S}^2) \cong C_0^\infty(\mathbb{S}^2) := \left\{ g \in C^\infty(\mathbb{S}^2) | \int g = 0 \right\}$$

- Vorticity $\omega = \nabla \times \mathbf{v} \cdot \mathbf{n}$:

$$\begin{cases} \dot{\omega} = \{\psi, \omega\} \\ \Delta\psi = \omega - f \end{cases}$$

where $\omega \in sdiff^*(\mathbb{S}^2)$, $f = (2\Omega \times \mathbf{v}) \cdot \mathbf{k}$ represents the **Coriolis force**

- $$H(\omega) = \frac{1}{2} \int \Delta^{-1}(\omega - f)(\omega - f),$$

Euler equations on a rotating sphere

- Work in a weak sense, extend *sdiff*$^*(\mathbb{S}^2)$ to $\mathcal{D}_0$
- $\omega \in \mathcal{D}_0 :=$ {distributions on $\mathbb{S}^2$ which integrate to 0}
- Existence of finite dimensional coadjoint orbits, i.e., point vortices (Marsden–Weinstein, 1983)

$$\omega_{PV} = \sum_{i=1}^{N_V} \Gamma_i \delta_{x_i} - \frac{\Gamma}{4\pi}$$

for $x_i \in \mathbb{S}^2$, $i = 1, \ldots, N_V$

- Splitting and restriction (Bogomolov, 1977):

$$\omega = \omega_{PV} \oplus \omega_S \in (\Gamma_i \delta^i_{\mathbb{S}^2})_{i=1,..,N_V} \bigoplus C_0^\infty(\mathbb{S}^2),$$

The coupled model (Bogomolov, 1977)

- Lie–Poisson Hamiltonian system on

$$\mathbb{R}^{3N_V} \bigoplus C_0^\infty(\mathbb{S}^2)$$

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$$H = -\frac{1}{8\pi} \sum_{\substack{i,j=1,..,N_V \\ i \neq j}} \Gamma_i \Gamma_j \log(1 - x_i \cdot x_j) + \\ + \sum_{i=1,..,N_V} \Gamma_i \Delta^{-1}(\omega - f)(x_i) + \\ + \frac{1}{2} \int \Delta^{-1}(\omega - f)(\omega - f).$$

The coupled model (Bogomolov, 1977)

$$\dot{x}_i = \frac{1}{4\pi} \sum_{\substack{j=1,..,N_V \\ i \neq j}} \Gamma_i \frac{x_i \times x_j}{1 - x_i \cdot x_j} + x_i \times \nabla \Delta^{-1}(\omega - f)(x_i)$$

$$\dot{\omega} = \{\omega, \Delta^{-1}(\omega - f) + \sum_{i=1,..,N_V} \Gamma_i \log(1 - x_i \cdot x)\},$$

for $i = 1, ..., N_V$

- Possibly replace $\log(1 - x \cdot y)$ with $\log(1 + \varepsilon - x \cdot y)$, for $\varepsilon > 0$

First integrals

- $H = H(x_1, \ldots, x_{N_V}, \omega)$
- $F_n(\omega) := \int \omega^n$, for $n = 2, 3, \ldots$
- $M(x_1, \ldots, x_{N_V}, \omega) := \sum_{i=1}^{N_V} \Gamma_i x_i + \begin{pmatrix} \int \omega x \\ \int \omega y \\ \int \omega z \end{pmatrix}$, if $\Omega = 0$

  otherwise $M \cdot \Omega$

Spatial discretization

- Geometric quantization of functions on the sphere (Hoppe, 1982):

$$C_0^\infty(\mathbb{S}^2, \mathbb{R}) \text{ is an } L_\alpha\text{-limit of } \mathfrak{su}(N) \text{ as } N \to \infty$$

- $$\omega_S = \sum_{\substack{l=1,..,\infty \\ m=-l,...,l}} \omega_S^{lm} Y_{lm} \xrightarrow{p_N} \sum_{\substack{l=1,..,N \\ m=-l,...,l}} \omega_S^{lm} T_{lm} =: W$$

  Y_{lm} spherical harmonics, $T_{lm} \in \mathfrak{su}(N)$

- Lie–Poisson system on $\mathbb{R}^{3N_V} \bigoplus \mathfrak{su}(N)$ w.r.t. $\times_{\mathbb{R}^3}^{N_V} \oplus [\cdot, \cdot]_N$

The coupled model

$$H = -\frac{1}{8\pi} \sum_{\substack{i,j=1,..,N_V \\ i \neq j}} \Gamma_i \Gamma_j \log(1 - x_i \cdot x_j) + $$

$$+ \sum_{\substack{i=1,..,N_V \\ l=1,..,N \\ m=-l,...,l}} \Gamma_i \frac{Y_{lm}(x_i)}{-l(l+1)} (W - COR)^{lm} + $$

$$+ \frac{1}{2} \mathrm{Tr}(\Delta^{-1} (W - COR)^T (W - COR)).$$

The coupled model

$$\dot{x}_i = \frac{1}{4\pi} \sum_{\substack{j=1,..,N_V \\ i \neq j}} \Gamma_i \frac{x_i \times x_j}{1 - x_i \cdot x_j} + \sum_{\substack{l=1,..,N \\ m=-l,...,l}} \frac{x_i \times \nabla Y_{lm}(x_i)}{-l(l+1)} (W - COR)^{lm}$$

$$\dot{W} = [\Delta^{-1} \Big(W + \sum_{\substack{i=1,..,N_V \\ l=1,..,N \\ m=-l,...,l}} \Gamma_i Y_{lm}(x_i) T_{lm} - COR \Big), W]_N.$$

for $i = 1, ..., N_V$

Discrete first integrals

- $H = H(x_1, \dots, x_{N_V}, W)$
- $F_n(W) := \mathrm{Tr}(W^n)$, for $n = 2, 3, \dots, N$
- $M(x_1, \dots, x_{N_V}, W) := \sum_{i=1}^{N_V} \Gamma_i x_i + \begin{pmatrix} \mathrm{Tr}(WT^x) \\ \mathrm{Tr}(WT^y) \\ \mathrm{Tr}(WT^z) \end{pmatrix}$,

  if $\Omega = 0$ otherwise $M \cdot \Omega$

Time discretization

- On $(\mathbb{S}^2)^{N_V}$ Spherical midpoint (Mclachlan, Modin, Verdier)
- On $\mathfrak{su}(N)$ Isospectral Runge-Kutta (Modin, V.)
- These provide a Lie–Poisson integrator on the global system

Idea for the time discretization

- $\mathfrak{su}(N) \cong \mathfrak{su}(N)^* \subset \mathfrak{gl}(N)^* \cong \mathfrak{gl}(N)$, via Frobenius norm
- Hamiltonian action of

$$SU(2)^{N_V} \times GL(N, \mathbb{C}) \text{ on } (\mathbb{C}^2)^{N_V} \times T^* GL(N, \mathbb{C})$$

- Equivariant momentum map

$$\mu : (\mathbb{C}^2)^{N_V} \times T^* GL(N, \mathbb{C}) \to (\mathfrak{su}(2)^*)^{N_V} \times \mathfrak{gl}(N, \mathbb{C})^*$$

- Trivial extension of H in the orthogonal complement of $\mathfrak{su}(N)^*$
- Using μ, lift the equations from

$$\mathbb{R}^{3N_V} \times \mathfrak{su}(N)^* \text{ to } (\mathbb{C}^2)^{N_V} \times T^* GL(N, \mathbb{C})$$

Idea for the time discretization

- Apply the implicit midpoint rule (symplectic integrator)
- Using μ, map back to $(\mathfrak{su}(2)^*)^{N_V} \times \mathfrak{gl}(N, \mathbb{C})^*$
- Descends to the Spherical midpoint on $\mathbb{R}^{3N_V}$ and to Isospectral midpoint rule on $\mathfrak{su}(N)^*$
- The final map is a Lie–Poisson integrator on $\mathbb{R}^{3N_V} \times \mathfrak{su}(N)^*$

Time discretization - Spherical midpoint

Theorem (McLachlan, Modin, Verdier, 2014)

Let $H \in C^\infty(\mathbb{R}^{3N_V})$, and let $\pi^* H$, with $\pi : \mathbb{C}^{2N_V} \to (su(2)^*)^{N_V} \cong \mathbb{R}^{3N_V}$ the momentum map corresponding to the canonical right action of $SU(2)^{N_V}$ on $\mathbb{C}^{2N_V}$, be the corresponding collective Hamiltonian on $\mathbb{C}^{2N_V}$. Then the implicit midpoint method applied to the Hamitonian vector field $X_{\pi^* H}$ descends to an integrator $\Phi_h(H)$ on $\mathbb{R}^{3N_V}$ which preserves the Lie–Poisson structure and the coadjoint orbits, i.e. the direct products of 2-spheres.

Spherical midpoint in $\mathbb{R}^{3N_V}$

Given $x_n \in \mathbb{R}^{3N_V}$,

$$x_{n+1} = x_n + hX_H\left(\frac{\sqrt{x_{n+1}^1 x_n^1}(x_{n+1}^1 + x_n^1)}{|x_{n+1}^1 + x_n^1|}, \ldots, \frac{\sqrt{x_{n+1}^{N_V} x_n^{N_V}}(x_{n+1}^{N_V} + x_n^{N_V})}{|x_{n+1}^{N_V} + x_n^{N_V}|}\right)$$

where $X_H(x) = x \times \nabla H(x)$ and the cross product is taken component-wise

Time discretization - Isospectral midpoint rule

Theorem (Modin, V., 2018)

Let $H \in C^\infty(\mathfrak{su}(N)^*)$, and let $\nu^* H$, with $\nu : T^*GL(N, \mathbb{C}) \to \mathfrak{gl}(N)^*$ the momentum map corresponding to the right cotangent lifted action of $GL(N, \mathbb{C})$ on $T^*GL(N, \mathbb{C})$, be the corresponding collective Hamiltonian on $T^*GL(N, \mathbb{C})$. Then the implicit midpoint method applied to the Hamitonian vector field $X_{\nu^* H}$ descends to an integrator $\Phi_h(H)$ on $\mathfrak{su}(N)^*$ which preserves the Lie–Poisson structure and the coadjoint orbits.

Isospectral midpoint rule in $\mathfrak{su}(N)^*$

Given $W_n \in \mathfrak{su}(N)^*$,

$$
\begin{cases}
X = -h(W_n + \frac{1}{2}X)\nabla H(W_n + \frac{1}{2}(X - X^T + K)) \\
K = \frac{h}{2}\nabla H(W_n + \frac{1}{2}(X - X^T + K))(X + K) \\
W_{n+1} = W_n + X - X^T + K - K^T
\end{cases}
$$

where the unknowns are X and K and the last line is explicit.

Time discretization - global method

Corollary

Let $H \in C^\infty(\mathbb{R}^{3N_V} \times \mathfrak{su}(N)^*)$, and let $\mu^* H$, with $\mu := (\pi, \nu)$ the momentum map as above defined component-wise. Then the implicit midpoint method applied to the Hamitonian vector field $X_{\mu^* H}$ descends to an integrator $\Phi_h(H)$ on $\mathbb{R}^{3N_V} \times \mathfrak{su}(N)^*$ which preserves the Lie–Poisson structure and the coadjoint orbits.

Properties of the - global method

- Intrinsically defined on $\mathbb{R}^{3N_V} \times \mathfrak{su}(N)^*$
- Lie–Poisson on $\mathbb{R}^{3N_V} \times \mathfrak{su}(N)^*$
  - Near conservation of the Hamiltonian $\sim \mathcal{O}(h^{-2})$
  - Round off precision conservation of the discrete Casimir functions F_n and momentum M (or M_z)
- $SU(2)^{N_V} \times SU(N)$-equivariant
- Second order of accuracy in time
- Second order of accuracy in space (quantization)

Relative equilibria

Definition (Relative equilibrium)

Let (M, ω, H) be and Hamiltonian system with symmetry group G. A point $x_0 \in M$ is said to be a relative equilibrium if there exists a 1-parameter subgroup $G_t \subset G$, such that

$$x(t) = \Phi_t^H(x_0) = g_t \cdot x_0$$

for some $\{g_t\}_{t\geq 0} \subset G_t$.

Relative equilibria for a fluid on a rotating sphere

- The group of symmetries is $SO(2)$
- In general, interaction between background vorticity and point vortices destroys non-rotating equilibria
- May stable non-rotating equilibria don't drift too far away from the initial position?

Numerical example (Laurent-Polz, Montaldi, Roberts, 2005)

A ring of 6 identical vortices, equally spaced on a latitudinal circle of co-latitude θ_0, is Lyapunov stable if

$$\cos^2\theta_0 > 4/5,$$

and linearly unstable if the inequality is reverse

First integrals variation

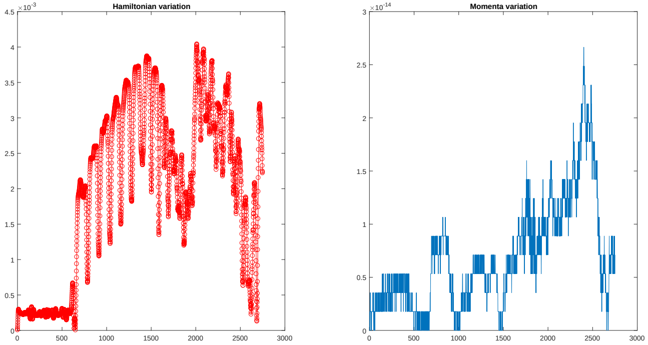

Figure: Hamiltonian and M_z variation; N=51, h=1e-3

Final conjecture

Conjecture

- *A stable $SO(2)$ relative equilibrium of point vortices persists when the sphere is rotating, whereas the background vorticity does not need to be in a relative equilibrium.*
- *Two polar vortices with $SO(2)$ symmetric initial smooth vorticity is the only global stable relative equilibrium of the coupled system on a rotating sphere.*

Thanks for your attention