Stable orbits in unstable systems: the case of the Pais-Uhlenbeck oscillator

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Introduction

- Renormalizability and unitarity are the main problems appearing in the quantization of fields
- There exist renormalizable theories based on higher order derivatives models, but most of them are discarded in Physics because of unitarity problems in their quantization
- Pais and Uhlenbeck developed one such model in the 50's, based on a generalization of the harmonic oscillator

Their model appeared in the context of a quantum theory of gravitation (as a toy model) and is described with the fourth-order differential equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}t^4} + (\omega_1^2 + \omega_2^2)\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega_1^2\omega_2^2 u = 0\,, \tag{1}$$

for a function of time $u = u(t)$. This is the Euler-Lagrange equation for the Lagrangian

$$L = \left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega_1^2 u\right)\left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \omega_2^2 u\right)\,.$$

The Ostrogadski's second-order formalism (generalization of the Legendre transform) allows one to find the corresponding Hamiltonian.

$$H_0 = \frac{1}{2}\left(p_1^2 + \omega_1^2 q_1^2\right) - \frac{1}{2}\left(p_2^2 + \omega_1^2 q_2^2\right) \,, \tag{2}$$

- Thus, the dynamical system defined by the fourth order equation can be described by the Hamiltonian which is the difference of two oscillators.
- Since two oscillators are uncoupled, there are no physical problems at the classical level.

For us, it is more important to consider the situation in which there is an interaction. If a nonlinear interaction term is added, say,

$$H = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) + \frac{\lambda}{4}(q_1 + q_2)^4 \,, \tag{3}$$

the interaction could lead to an exchange of energy from the 'positive' oscillator to the 'negative' one, and this exchange could be done without any lower bound.

An infinite amount of energy could be dissipated by the 'negative' oscillator, collapsing the system.

The situation just described has been the main reason to discard higher-order models as useful physical models.

However, their property of being perturbatively renormalizable is strong enough to have prevented the complete vanishing of interest in them.

- Some numerical studies have been conducted with the hope of finding a mechanism that would protect unitarity or prevent the collapse of the system.
- Surprinsingly, the numerical experiments have found the existence of a class of interaction admitting **"islands of stability"** (Smilga 2009, Pavsic 2013).

For some values of the parameters characterizing the system, the interaction generates **stable periodic motions,** thus allowing these models to be considered as perfectly viable ones.

Some **heuristic arguments have been given** in these papers to justify the occurrence of these stable motions. But this phenomenon continues to be basically a finding based on numerical simulations, and **its physical origin remains unexplained**.

Our purpose: To give a mathematical proof of the existence of stable closed orbits for the system described by the Hamiltonian function

$$H = H_0 + \lambda H_1 = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) + \frac{\lambda}{4}(q_1 + q_2)^4$$

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Tools: No need to reinvent the wheel. Very classical ones!

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2. Hamiltonian systems with symmetries and their reduced phase spaces
3. Poincaré sections

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However things are not straightforward, as the usual theorems do not apply!

The whole idea:

1. The complete Hamiltonian

$$H = H_0 + \lambda H_1 = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) + \frac{\lambda}{4}(q_1 + q_2)^4$$

   is regarded as a perturbation of the Pais-Uhlenbeck oscillator. The Hamiltonian flow of H_0 is periodic.

2. Using normal form theory (via averaging method), we analyze the normal form of H on the reduced phase space defined by the $\mathrm{U}(1)-$symmetry of H_0.
3. Look for periodic orbits which are preserved under perturbation.
4. In general, the orbit space of a regular level value of H_0 has singularities. We look for periodic orbits by studying a suitable first return map of a Poincaré section.

Generators of symmetry algebra of H_0

- In the phase space $\mathbb{R}^4 = \{(p_1, q_1, p_2, q_2) | p_1, q_1, p_2, q_2 \in \mathbb{R}\}$, consider the usual Poisson bracket in $\mathcal{C}^\infty(\mathbb{R}^4)$ defined by

$$\{f, g\} := \sum_{i=1}^{2} \frac{\partial f}{\partial p_i} \cdot \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \cdot \frac{\partial g}{\partial p_i}$$

- Symmetry algebra of the Pais-Uhlenbeck oscillator

$$\mathcal{A}_{H_0} = \{F \in C^\infty | \{H_0, F\} = 0\}$$

The algebra $\mathcal{A}_{H_0}$ is a finitely-generated $\mathbb{R}$-algebra (generators are polynomial functions), and it is a Poisson subalgebra of $(C^\infty(\mathbb{R}^4), \{\ ,\})$.

A good description of the normal forms and the reduced space is obtained using the generators of $\mathcal{A}_{H_0}$

Introduce the complex coordinates

$$z_j = p_j + i\omega_j q_j, \quad \overline{z}_j = p_j - i\omega_j q_j\,, \quad j = 1, 2.$$

The Pais-Uhlenbeck oscillator takes the form

$$H_0 = \frac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2).$$

The Poisson bracket is given by

$$\{f, g\} = 2\mathrm{i}\sum_{i=1}^{2}\omega_i\left(\frac{\partial f}{\partial \bar{z}_i}\frac{\partial g}{\partial z_i} - \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \bar{z}_i}\right).$$

Consider the homogeneous monomials

$$P = z_1^{k_1^+} \bar{z}_1^{k_1^-} z_2^{k_2^+} \bar{z}_2^{k_2^-}. \tag{4}$$

We observe that

$$\{H_0, P\} = \mathrm{i}\left[\omega_1(k_1^+ - k_1^-) - \omega_2(k_2^+ - k_2^-)\right] P. \tag{5}$$

Therefore, $P \in \mathcal{A}_{H_0}$ if and only if

$$\omega_1 r - \omega_2 s = 0, \tag{6}$$

where $r = k_1^+ - k_1^-, s = k_2^+ - k_2^-$.
If (r, s) is an integer solution, then there exist an integer k such that $(r, s) = (k\omega_2, k\omega_1)$

The set of solutions form a monoid with respect to the sum of vectors in $\mathbb{R}^2$. The generator of this monoid are given by the vector solutions

$$(0,0), \quad (\omega_2, \omega_1) \text{ and } (-\omega_2, -\omega_1).$$

These vector solutions correspond to the homogeneous monomials $P = z_1^{k_1^+} \bar{z}_1^{k_1^-} z_2^{k_2^+} \bar{z}_2^{k_2^-}$ satisfying

(a) $k_1^+ - k_1^- = 0$, $k_2^+ - k_2^- = 0$,
(b) $k_1^+ - k_1^- = \omega_2$, $k_2^+ - k_2^- = \omega_1$,
(c) $k_1^+ - k_1^- = -\omega_2$, $k_2^+ - k_2^- = -\omega_1$,

Taking into account that the power k_i^+, k_i^- are non-negative, the generators of the symmetry algebra turn out to be

$$\begin{aligned}
\rho_1 &= z_1\bar{z}_1, \\
\rho_2 &= z_2\bar{z}_2 \\
\rho_3 &= \mathrm{Re}\left(z_1^{\omega_2} z_2^{\omega_1}\right), \\
\rho_4 &= \mathrm{Im}\left(z_1^{\omega_2} z_2^{\omega_1}\right),
\end{aligned}$$

These generators satisfy the following relation

$$\rho_1^{\omega_2}\rho_2^{\omega_1} = \rho_3^2 + \rho_4^2.$$

The realization of the generators in momentum-position coordinates are

$$\begin{aligned}
\rho_1 &= (p_1^2 + \omega_1^2 q_1^2), \\
\rho_2 &= (p_2^2 + \omega_2^2 q_2^2) \\
\rho_3 &= \mathrm{Re}\left((p_1 + i\omega_1 q_1)^{\omega_2}(p_2 + i\omega_2 q_2)^{\omega_1}\right), \\
\rho_4 &= \mathrm{Im}\left((p_1 + i\omega_1 q_1)^{\omega_2}(p_2 + i\omega_2 q_2)^{\omega_1}\right),
\end{aligned}$$

The commutation relations between generators read

$$\begin{aligned} \{\rho_1, \rho_2\} &= 0, \quad \{\rho_1, \rho_3\} = -2\omega_2\omega_1\rho_4, \quad \{\rho_1, \rho_4\} = 2\omega_2\omega_1\rho_3, \\ \{\rho_2, \rho_3\} &= -2\omega_2\omega_1\rho_4, \quad \{\rho_2, \rho_4\} = 2\omega_2\omega_1\rho_3, \\ \{\rho_3, \rho_4\} &= \omega_2\omega_1\rho_1^{\omega_2-1}\rho_2^{\omega_1-1}(\omega_1\rho_1 + \omega_2\rho_2). \end{aligned} \tag{7}$$

Moreover, every $f \in \mathcal{A}_{H_0}$ commutes with the function

$$F = \rho_1^{\omega_2}\rho_2^{\omega_1} - \rho_3^2 + \rho_4^2\,.$$

Normal forms

- We say that the perturbed Hamiltonian $H_0 + \lambda G + O(\lambda^2)$ is in normal form to first order wrt H_0 if $\{H_0, G\} = 0.$

- For the Hamiltonian $H = H_0 + \lambda H_1$, there exists a near identity transformation $\mathcal{T}_\lambda : N \subset \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\mathcal{T}_\lambda^* H = H_0 + \lambda \langle H_1 \rangle + O(\lambda^2),$$

where

$$\langle H_1 \rangle = \frac{1}{2\pi\omega_1\omega_2} \int_0^{2\pi\omega_1\omega_2} (\mathrm{Fl}^{\mathrm{t}}_{\mathrm{X}_{\mathrm{H}_0}})^* H_1 \, \mathrm{d}t \, .$$

The computation of the first-order normal form of

$$H_0 + \lambda H_1 = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2) + \frac{\lambda}{4}(q_1 + q_2)^4 \,,$$

is given by the averaging of the perturbation term H_1 along the flow $\mathrm{Fl}^t_{X_{H_0}}$, that is:

$$\langle H_1 \rangle = \frac{1}{2\pi\omega_1\omega_2} \int_0^{2\pi\omega_1\omega_2} (\mathrm{Fl}^{\mathrm{t}}_{\mathrm{X_{H_0}}})^* H_1 \, \mathrm{d}t \,.$$

Normal forms

The flow of X_{H_0} is given by

$$\mathrm{Fl}^t_{X_{H_0}}\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{p_1}{\omega_1}\sin\omega_1 t + q_1\cos\omega_1 t \\ p_1\cos\omega_1 t - \omega_1 q_1 \sin\omega_1 t \\ q_2\cos\omega_2 t - \frac{p_2}{\omega_2}\sin\omega_2 t \\ \omega_2 q_2 \sin\omega_2 t + p_2\cos\omega_2 t \end{pmatrix}. \tag{8}$$

the flow is periodic with constant period

$$T = 2\pi\omega_1\omega_2$$

$$
\begin{aligned}
\langle H_1 \rangle &= \frac{3}{32\omega_1^4\omega_2^4}\left(q_1^2(4\omega_1^4\omega_2^4 q_2^2 + 2\omega_1^2\omega_2^4 p_1^2 + 4\omega_1^4\omega_2^2 p_2^2)\right. \\
&\quad +\omega_1^4\omega_2^4(q_1^4+q_2^4) + q_2^2(4\omega_1^2\omega_2^4 p_1^2 + 2\omega_1^4\omega_2^2 p_2^2) \\
&\quad \left.+p_1^2(\omega_2^4 p_1^2 + 4\omega_1^2\omega_2^2 p_2^2) + \omega_1^4 p_2^4.\right)
\end{aligned}
$$

Since $\{H_0, H_0 + \lambda \langle H_1 \rangle\} = 0$, we re-express these results in terms of the Hopf invariants ρ_i, obtaining

$$
H_0(\rho_1, \rho_2, \rho_3, \rho_4) = \frac{1}{2}(\rho_1 - \rho_2) \tag{9}
$$

for the free part, while

$$
\langle H_1 \rangle (\rho_1, \rho_2, \rho_3, \rho_4) = \frac{3}{8}\left(\frac{1}{4\omega_1^4}\rho_1^2 + \frac{1}{\omega_1^2\omega_2^2}\rho_1\rho_2 + \frac{1}{4\omega_2^4}\rho_2^2\right). \tag{10}
$$

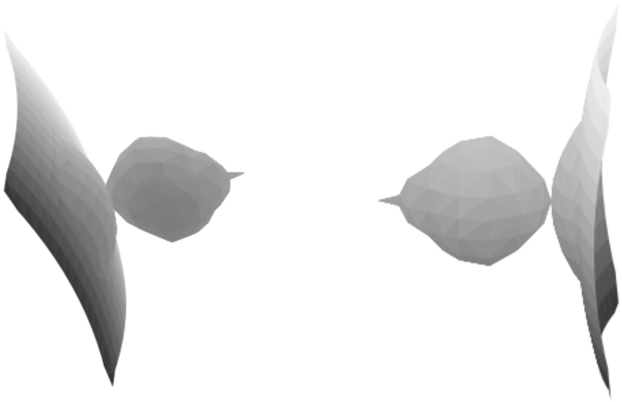
A model for the reduced space

At this point, there are two things to do:

1. Obtain a good model for the reduced phase space,
2. Give an explicit expression for the reduced Hamiltonian, that is, the normal form Hamiltonian $N = H_0 + \lambda N_1 + O(\lambda^2)$ when restricted to the reduced phase space.

We will restrict our attention to fixed negative energy values. The reduced phase space is then given by the set of equations

$$\begin{cases} \rho_3^2 + \rho_4^2 = \rho_1^{\omega_2} \rho_2^{\omega_1}\,, \quad \rho_1 \geq 0,\, \rho_2 \geq -2h,, \\ \rho_1 - \rho_2 = 2h\,, \end{cases}$$



Rename the variables $\rho_3 = x$, $\rho_4 = y$, and $\rho_2 = z$. The commutation relations allows to define a Poisson bracket on $\mathbb{R}^3 = \{(x, y, z)\}$:

$$\{f, g\} = \omega_1 \omega_2 < \nabla g, \nabla f \times \nabla F >, \tag{11}$$

where F is the function

$$F(x, y, z) = x^2 + y^2 - (z + 2h)^{\omega_2} z^{\omega_1} \tag{12}$$

If we define

$$\begin{aligned} \Psi \quad &: \quad \mathbb{R}^4 \to \mathbb{R}^3, \\ \Psi(p_1, q_1, p_1, q_1) \quad &= \quad (\rho_3, \rho_4, \rho_2) \end{aligned}$$

we see that Ψ is a Poisson mapping and $\Psi(H_0^{-1}(h)) = F^{-1}(0)$.

- Moreover,

$$\Psi(\mathrm{Fl}^t_{X_{H_0}}\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}) = \Psi(\rho_1(p,q), \rho_2(p,q), \rho_3(p,q), \rho_4(p,q)).$$

- Therefore, the reduced space is contained in a symplectic leaf of $F^{-1}(0) \subset \mathbb{R}^3$. Let us denote by M_h the reduced space.

$$M_h = \begin{cases} F^{-1}(0) \text{ and } z \geq -2h & \text{if } \omega_2 = 1, \\ F^{-1}(0) \text{ and } z > -2h & \text{if } \omega_2 > 1. \end{cases} \tag{13}$$

- Notice that the point $(x, y, z) = (0, 0, -2h)$ is always a point on $F^{-1}(0)$ and this manifold is smooth at this point only when $\omega_2 = 1$.
- The point corresponds to the curve in $\mathbb{R}^4$ given by $p_1 = q_1 = 0$ and $p_2 + \omega_2^2 q_2 = -2h$.

- Any function $f \in C^\infty(\mathbb{R}^3)$ defines a Hamiltonian vector field $\widetilde{X}_f$ on M_h given by

$$\widetilde{X}_f := (\omega_1\omega_2 \nabla f \times \nabla F)|_{M_h}.$$

$\widetilde{X}_f$ has a critical point at the point $p \in M_h$ if and only if $\nabla f(p)$ is orthogonal at p to reduced space M_h or $\nabla f(p) = 0$.

- We consider the function $N_1(\rho_1, \rho_2, \rho_3, \rho_4)$ (10). Let $K(x, y, z) = N_1(z + 2h, z, x, y)$,

$$K(x, y, z) \;=\; \frac{3}{8}\left(\frac{1}{4\omega_1^4}(z+2h)^2 + \frac{1}{\omega_1^2\omega_2^2}(z+2h)z + \frac{1}{4\omega_2^4}z^2\right).$$

- By straight forward computation, we get

$$\nabla K = (0, 0, \frac{3}{8}\left(\frac{\omega_1^4 + 4\omega_1^2\omega_2^2 + \omega_2^4}{2\omega_1^4\omega_2^4} z + \frac{2\omega_1^2 + \omega_2^2}{\omega_1^4\omega_2^2} h\right)).$$

Thus, $\nabla K(p) = 0$, if and only if $p = (0, 0, (-2h)\frac{2\omega_1^2\omega_2^2+\omega_2^4}{\omega_1^4+4\omega_1^2\omega_2^2+\omega_2^4})$.

- Because of $\frac{2\omega_1^2\omega_2^2+\omega_2^4}{\omega_1^4+4\omega_1^2\omega_2^2+\omega_2^4} < 1$, ∇K never vanishes on M_h.
- Since ∇K never vanishes on M_h, ∇K is orthogonal to M_h at $(0, 0, -2h)$ if and only if $\omega_2 = 1$ (otherwise this point does not belong to M_h.)

The case $\omega_2 = 1$.

- **Moser's theorem:** the existence of non-degenerate critical points of $\widetilde{X}_K$ implies the existence of periodic orbits for the PU-oscillator.

- Since $\frac{\partial F}{\partial z}(0, 0, -2h) = (-2h)^{\omega_1} \neq 0$, there there exists a smooth function (locally defined) $z = g(x, y)$ such that $F(x, y, g(x, y)) = 0$ and $g(0, 0) = -2h$.

- Then, $\widetilde{K} = K(g(x, y))$ in a neighborhood of $(0, 0, -2h)$. A direct computation shows
$$\mathrm{Hess}(\widetilde{K}(0, 0)) > 0.$$

The case $\omega_2 > 1$.

- If $\omega_2 > 1$ neither ∇K vanishes on M_h nor ∇K is orthogonal to M_h: Moser's theorem does not apply.

- Recall that we have removed the point $(0, 0, -2h)$, corresponding to the curve

$$\gamma = \{(p_1, q_1, p_2, q_2) | p_1 = q_1 = 0, \quad p_2 + \omega_2^2 q^2 = -2h\}.$$

- Let $f_2(p_1, q_1, p_2, q_2) = \frac{1}{2}(p_2 + \omega_2^2 q^2)$. The Hamiltonian vector field X_{f_2} has periodic flow with period $T = \frac{2\pi}{\omega_2}$.

- For every fixed $h < 0$, the level set $f_2^{-1}(-h)$ is foliated by periodic orbits of X_{f_2} and a the reduced space is given by $M_h = f_2^{-1}(-h)/\mathbb{S}^1$.

- Make the following change of variables Ψ on $\mathbb{R}^2 \times (\mathbb{R}^2 - (0,0))$, with $L > 0$ and $0 < \theta < \frac{2\pi}{\omega_2}$:

$$\Psi(p_1, q_1, p_2, q_2) = \left(p_1, q_1, -\sqrt{2L}\sin\omega_2\theta, -\frac{\sqrt{2L}}{\omega_2}\cos\omega_2\theta\right)$$

- The canonical simplectic form $\mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2$ takes the form $\mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}\theta \wedge \mathrm{d}L$ and the Hamiltonian of Pais-Uhlenbeck oscillator can be rewritten as

$$H(p_1, q_1, L, \theta) = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - L + \frac{\lambda}{4}(q_1 - \frac{\sqrt{2L}}{\omega_2}\cos\omega_2\theta)^4.$$

- We restrict to the level set $\Sigma_h := \{(p_1, q_1, L, \theta) | L = h\}$.

- the Hamiltonian equations on Σ_h are

$$\begin{aligned}\dot{\theta} &= 1 + \lambda(q_1 - \frac{\sqrt{-2h}}{\omega_2}\cos\omega_2\theta)^3 \left(\frac{\cos\omega_2\theta}{\omega_2\sqrt{-2h}}\right), \\ \dot{p}_1 &= -\omega_1^2 q_1 - \lambda(q_1 - \frac{\sqrt{-2h}}{\omega_2}\cos\omega_2\theta)^3, \\ \dot{q}_1 &= p_1.\end{aligned} \tag{14}$$

- We consider the cross section $\sigma_0 = \{(p_1, q_1, -h, \theta) \in \Sigma_h | \theta = 0\}$.

- We fix $a = ((p_1^0, q_1^0, -h, 0)) \in \sigma_0$. Then

$$\begin{aligned}
\theta(t) &= t + \lambda \int_0^t (q_1 - \frac{\sqrt{-2h}}{\omega_2}\cos\omega_2\theta)^3 \left(\frac{\cos\omega_2\theta}{\omega_2\sqrt{-2h}}\right) dt, \\
p_1(t) &= p_1^0 \cos\omega_1 t - \omega_1 q_1^0 \sin\omega_1 t - \lambda \int_0^t (q_1 - \frac{\sqrt{-2h}}{\omega_2}\cos\omega_2\theta)^3 dt, \\
q_1(t) &= \frac{p_1^0}{\omega_1}\sin\omega_1 t + q_1 \cos\omega_1 t.
\end{aligned}$$

- Let $T(a,\lambda)$ be the elapsed time between two consecutive intersection of σ_0. From equation (15), we have

$$\frac{2\pi}{\omega_2} = T(a,\lambda) + \lambda \int_0^{T(a,\lambda)} (q_1 - \frac{\sqrt{-2h}}{\omega_2}\cos\omega_2\theta)^3 \left(\frac{\cos\omega_2\theta}{\omega_2\sqrt{-2h}}\right) dt.$$

- Then, $T(a,\lambda)$ has the form $T(a,\lambda) = 2\pi + \lambda T_1(a) + O(\lambda^2)$, where

$$T_1(a) = \frac{1}{\omega_2\sqrt{-2h}} \int_0^{2\pi/\omega_2} \cos\omega_2 t \left(\frac{\sqrt{-2h}}{\omega_2} \cos\omega_2 t - q_1^0 \right)^3$$

(with $T_1(0,0,-h,0) \neq 0$)

- Substituting, we obtain

$$p_1(T) = p_1^0 + \lambda \left(-\omega_1^2 q_1^0 T_1(a) - \int_0^{\frac{2\pi}{\omega_2}} (q_1^0 - \frac{\sqrt{-2h}}{\omega_2} \cos\omega_2 t)^3 dt \right) + O(\lambda^2)$$
$$q_1(T) = q_1^0 + \lambda p_1^0 T_1(a) + O(\lambda^2).$$

- There will be periodic orbits for PU-oscillator in Σ_h if there exist $p_1^0(\lambda)$ and $q_1^0(\lambda)$ such that

$$p_1(T(p_1^0(\lambda), q_1^0(\lambda), -h, 0, \lambda)) = p_1^0(\lambda),$$
$$q_1(T(p_1^0(\lambda), q_1^0(\lambda), -h, 0, \lambda)) = q_1^0(\lambda).$$

- Define the function $F : \mathbb{R}^3 \to \mathbb{R}^2$

$$F(p_1, q_1, \lambda) = \begin{pmatrix} -\omega_1^2 q_1 T_1(a) - \int_0^{2\pi/\omega_2} (q_1 - \frac{\sqrt{-2h}}{\omega_2} \cos\omega_2 t)^3 dt + O(\lambda) \\ p_1 T_1(a) + O(\lambda) \end{pmatrix}$$

- By a direct computation, we note that $F(0,0,0) = (0,0)^T$ and

$$\det \left(\frac{\partial F}{\partial p_1 \partial q_1} \Big|_{(0,0,0)} \right) =$$
$$\det \begin{bmatrix} 0 & -\omega_1^2 T_1(0,0,-h,0) \\ T_1(0,0,-h,0) & 0 \end{bmatrix} > 0\,.$$

- By the implicit function theorem, there exists $\delta > 0$, and open neighborhood U of $(0,0)$ and a function $g : (-\delta, \delta) \to U$, $g(\lambda) = (p_1(\lambda), q_1(\lambda))$ such that $g(0) = (0,0)$ and $F(g(\lambda), \lambda) = 0$. Therefore,

$$p_1(T(g(\lambda), -h, 0, \lambda)) = p_1(\lambda),$$
$$q_1(T(g(\lambda), -h, 0, \lambda)) = q_1(\lambda).$$

- This fact proves that for small enough λ, the PU-oscillator has a stable periodic orbit γ_λ with energy h which arises from the normal mode γ.