GROUP-VALUED MOMENTUM MAPS FOR ACTIONS OF AUTOMORPHISM GROUPS

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ROUGH OVERVIEW OF THE RESULTS

• The space of smooth sections of a fiber bundle whose fibers are symplectic manifolds (called *symplectic fiber bundle*) carries a natural symplectic structure. The group of bundle automorphisms acts symplectically on this space but does not admit a classical momentum map.

• We introduce a new concept of a group-valued momentum map, inspired by the Poisson Lie setting. The group-valued momentum map assigns to every section of the symplectic fiber bundle a principal circle-bundle. We study the properties of this group-valued momentum map.

• Many examples can be handled with this new momentum map:

◇ obtain generalized Clebsch variables for fluids with integral helicity;

◇ the anti-canonical bundle is the momentum map for the action of symplectomorphisms on the space of compatible complex structures;

◇ the Teichmüller moduli space is realized as a symplectic orbit reduced space associated to a coadjoint orbit of $SL(2, \mathbb{R})$ and spaces related to the other coadjoint orbits are identified and studied;

◇ the momentum map for the group of bundle automorphisms on the space of connections over a Riemannian surface encodes, besides the curvature, also topological information of the bundle.

DETAILED OVERVIEW OF THE RESULTS

Noether's theorem states that every symmetry of a system generates a conservation law. In symplectic geometry formulation, these conserved quantities are encoded in the momentum map.

The momentum map is not only important in dynamical systems but is also a valuable tool in the study of differential geometric questions. Atiyah and Bott [1983] showed that the curvature of a connection on a principal bundle over a Riemann surface furnishes the momentum map for the action of the group of gauge transformations. They applied Morse theory to the norm-squared of the momentum map (the Yang–Mills functional) in order to obtain the cohomology of the moduli space of Yang–Mills solutions which, by the Narasimhan–Seshadri theorem, can be identified with the moduli space of stable holomorphic structures.

Later, Fujiki [1992] and Donaldson [1997, 1999, 2003] provided a momentum map picture for the relationship between the existence of constant scalar curvature Kähler metrics and stability in the sense of geometric invariant theory.

FIRST GOAL: Provide a framework which encompasses the gauge theory setting of Atiyah and Bott [1983] together with the action of diffeomorphism groups of Fujiki [1992] and Donaldson [1997].

Starting point is a symplectic fiber bundle of the form $F = P \times_G \underline{F}$ for a principal G-bundle $P \to M$, where the typical fiber $\underline{F}$ is endowed with a G-invariant symplectic form. The fiberwise symplectic structure, combined with a volume form on the base M, induces a symplectic form Ω on the space $\mathscr{F}$ of sections of $F \to M$. The gauge group of P acts in a natural way on $\mathscr{F}$, leaving the induced symplectic form Ω invariant.

We show that the action possesses a momentum map which is completely determined by the momentum map of the G-action on the fiber $\underline{F}$.

If the bundle P is natural, i.e., it comes with a lifted Diff(M)-action to bundle automorphisms (for example, this is the case when P is the frame bundle of M), then the group of volume-preserving diffeomorphisms acts on the space $\mathscr{F}$ of sections and leaves Ω invariant. A precise statement will be given later.

There are essentially two contributions to the momentum map. The first term is the pull-back of the fiberwise symplectic structure. The second term involves the fiber momentum map and, morally speaking, captures how much the lift of diffeomorphisms to bundle automorphisms shifts in the vertical direction. The interesting point is that the momentum map for the automorphism group on the *infinite-dimensional* space of sections is canonically constructed from the *finite-dimensional* symplectic G-manifold $\underline{F}$.

In contrast to the case of the action of the gauge group, the momentum map for the symplectic action of the diffeomorphism group on the space of sections *does not exist in full generality*. Donaldson [2000, 2003] already pointed this out. The obstruction has a topological character, i.e., certain cohomology groups have to vanish. To remedy this situation, one usually restricts the actions to certain "exact" subgroups, e.g., the subgroup of Hamiltonian diffeomorphisms in the group of all symplectomorphisms. Then, the actions of these subgroups do admit classical momentum maps.

Working from a completely different point of view, similar observations were made by Gay-Balmaz and Vizman [2012] in their study of the classical dual pair in hydrodynamics. In this case, the symplectic action of volume-preserving diffeomorphisms on a symplectic manifold of mappings only has a momentum map under certain topological conditions and one is forced to work with suitable central extensions of the group of exact volume-preserving diffeomorphisms.

OUR POINT OF VIEW: The above mentioned topological obstructions are not a bug but a feature of the theory. The action of the diffeomorphism group interacts with, and is largely determined by, the topological structure of the bundle. Thus, one would expect to capture certain topological data (like characteristic classes) that are "conserved" by the action and such "conservation laws" should be encoded in the momentum map. Since the classical momentum map takes values in a continuous vector space, there is no space to "store" discrete topological information. Hence, whenever those classes do not vanish, the momentum map does not exists.

SECOND GOAL: Turn these philosophical remarks into explicit mathematical statements. In order to do this, we generalize the notion of momentum maps allowing them to take values in groups.

Our concept of a group-valued momentum map is inspired by the Poisson geometric Lu momentum map introduced in her 1990 thesis and in Lu and Weinstein [1990]. The group-valued momentum map introduced here is a vast generalization of many notions of momentum maps appearing in the literature including circle-valued, cylinder-valued, and Lie algebra-valued momentum maps. We show that our generalized group-valued momentum map always exists for the action of the diffeomorphism group, without any topological assumptions on the base but some integrability conditions on the fiber model. The resulting momentum map captures topological invariants of the geometry, exactly in (the dual of) those cohomology classes which prevented the existence of a classical momentum map. This approach of extending the definition of the momentum map, besides the situation described above in Poisson geometry, in order to capture conservation laws not available using the classical definition, has been used successfully before in the theory of the cylinder-valued and optimal momentum maps (Ortega and Ratiu [2003]).

Hydrodynamic example: Let (M,μ) be a closed (i.e., compact connected, boundaryless) n-manifold with volume form μ (identified with the measure it defines, also denoted by μ) and (F,ω) a symplectic manifold. The space $C^\infty(M,F)$ of smooth maps from M to F carries the weak symplectic form

$$\Omega_\phi(X,Y) = \int_M \omega_{\phi(m)}(X(m),Y(m))\, d\mu(m),$$

where $\phi \in C^\infty(M,F)$ and $X,Y \in \mathsf{T}_\phi C^\infty(M,F)$, i.e., $X,Y : M \to \mathsf{T}F$ satisfy $X(m),Y(m) \in \mathsf{T}_{\phi(m)}F$ for all $m \in M$. The natural action by precomposition of $\mathrm{Diff}_\mu(M)$, the group of diffeomorphisms of M preserving the volume form μ, leaves Ω invariant. If ω is exact, say with primitive ϑ, then the momentum map assigns the 1-form $\phi^*\vartheta$ to a map $\phi \in C^\infty(M,F)$. Here, the space of volume-preserving vector fields $\mathfrak{X}_\mu(M)$ (the vector fields whose μ-divergence vanishes) is identified with closed $(n-1)$-forms, i.e., $\mathfrak{X}_\mu(M)^* = \Omega^1(M)/\mathrm{d}\Omega^0(M)$. More generally, Gay-Balmaz and Vizman [2012] showed that a (non-equivariant) momentum map also exists when the pull-back of ω by all maps $\phi \in C^\infty(M,F)$ is exact; for example, this happens when $\mathrm{H}^2(M)$ is trivial.

Our generalized group-valued momentum map takes no longer values in $\mathfrak{X}_\mu(M)^*$, but instead in the Abelian group $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ that parametrizes principal circle bundles with connections modulo gauge equivalence. If (F, ω) has a prequantum bundle (L, ϑ), then our group-valued momentum map sends ϕ to the pull-back bundle with connection $\phi^*(L, \vartheta)$. We see that no (topological) restrictions have to be made for M and only the integrability condition of the symplectic form ω is needed for the existence of a group-valued momentum map. In contrast to the classical momentum map, a $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$-valued momentum map contains topological information. First, the Chern class of the bundle, as a class in $\mathsf{H}^2(M, \mathbb{Z})$, is available from the generalized momentum map. In our simple example, this is just the integral refinement of $\phi^*\omega$. A second class in $\mathsf{H}^1(M, \mathsf{U}(1))$ is related to the equivariance of the momentum map; we will make all of this precise later on.

	Space	Action of	Chern class	Secondary topological class
Hydrodynamics	$C^\infty(M, F)$	$\mathrm{Diff}_\mu(M)$	0 (total vorticity) in $H^2(M, \mathbb{Z})$	Circulations in $H^1(M)$
Lagrangian embeddings	$C^\infty(L, M)$	$\mathrm{Diff}_\mu(L)$	Torsion class in $H^2(M, \mathbb{Z})$	Liouville class in $H^1(M)$
Kähler geometry	$\Gamma^\infty(\mathrm{L}M \times_{\mathrm{Sp}} \mathrm{Sp}/\mathrm{U})$	$\mathrm{Diff}_\omega(M)$	$c_1(M) \cup [\omega]^{n-1}$ in $H^{2n}(M, \mathbb{Z})$	
Quantomorphism	$\Gamma^\infty(P \times_{\mathrm{U}(1)} \mathbb{C})$	$\mathrm{Aut}_\Gamma(P)$	trivial	
Gauge theory	$\mathrm{Conn}(P)$	$\mathrm{Aut}(P)$	Torsion class in $H^{2n}(M, \mathbb{Z})$	

Overview of the examples. Here, μ is a volume form and ω a symplectic form. Moreover, $Q \to M$ denotes a prequantum circle bundle with connection Γ and $P \to M$ is an arbitrary principal G-bundle. The frame bundle is denoted by $\mathrm{L}M$. We also abbreviated the homogeneous space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ by Sp/U.

Comments on the examples in the table

1.) Marsden and Weinstein [1983] construct Clebsch variables for ideal fluids starting from a similar infinite-dimensional symplectic system as discussed above. It turns out, that every vector field represented in those Clebsch variables has vanishing helicity, i.e., such a fluid configuration has trivial topology and no links or knots. Our more general framework allows to construct generalized Clebsch variables for vector fields with integral helicity.

2.) When applied to the space of Lagrangian immersions, the group-valued momentum map recovers the Liouville class as the topological data. Thus, we realize moduli spaces of Lagrangian immersions (and modifications thereof) as symplectic quotients.

3.) and 4.) Many examples with geometric significance are obtained when the typical fiber $\underline{F}$ is a symplectic homogeneous space G/H. In this case, sections of $\mathsf{L}M \times_G \underline{F}$ correspond to a reduction of the G-frame bundle $\mathsf{L}M$ to H.

Important case: the space of almost complex structures compatible with a given symplectic structure, i.e., $\underline{F} = \mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n)$. In this case, the group-valued momentum map for the group of symplectomorphisms assigns to an almost complex structure the anti-canonical bundle. It was already observed by Fujiki [1992] and Donaldson [1997] that the Hermitian scalar curvature furnishes a classical momentum map for the action of the group of Hamiltonian symplectomorphisms. Of course, the Hermitian scalar curvature is the curvature of the anti-canonical bundle. Thus the group-valued momentum map combines the geometric curvature structure with the topological data of the anti-canonical bundle. For the case of a 2-dimensional base manifold, we realize the Teichmüller moduli space with the symplectic Weil–Petersson form as a symplectic orbit reduced space.

5.) We extend the classical setting of Atiyah and Bott [1983] in two ways. First, we generalize the gauge theoretic setting from 2-dimensional surfaces to arbitrary symplectic manifolds as the base (a similar extension was already discussed by Donaldson [1987]).

Secondly, and more importantly, we determine the group-valued momentum map for the action of the *full automorphism group* on the space of connections. Besides the curvature, the group-valued momentum map also encodes a torsion class in $\mathsf{H}^{2n}(M,\mathbb{Z})$, that arises from flat group homomorphisms $\mathsf{Sp}(2n,\mathbb{R}) \times G \to \mathsf{U}(1)$.

General comments

1.) In contrast to most papers discussing infinite-dimensional symplectic geometry, we do not work formally, but really address the functional analytical problems arising from the transition to the infinite-dimensional setting. In particular, smoothness of maps between infinite dimensional manifolds is understood in the sense of locally convex spaces as, for example, presented in Neeb [2006] or the Russian school (Smolyanov).

2.) Integrality of certain symplectic forms play a central role. To a large extent, this assumption was made for convenience. Most results carry directly over to symplectic forms with discrete period groups per$\omega \subseteq \mathbb{R}$, without much technical effort. In spirit, our results also hold in the general setting without any assumptions on the period group; however, then one is forced to work in the diffeological category because the quotient $\mathbb{R}/\text{per}\omega$ may no longer be a Lie group.

3.) Most of our symplectic reduced spaces are obtained as (sometimes singular) *orbit* reduced spaces, a theory that is not yet present in the literature for infinite dimensional systems, even though we state theorems using it. However, the techniques in Diez [2018] which completely treats infinite dimensional singular symplectic *point* reduction, combined with the strategy in the book by Ortega and Ratiu [2003] for finite dimensional singular symplectic orbit reduction, yields a general theory of infinite dimensional singular symplectic orbit reduction, which is precisely what is needed here. This is the focus of a future paper.

GROUP-VALUED MOMENTUM MAPS

Inspiration is the momentum map for Poisson Lie group actions. Our group-valued momentum map is *not* built on the pattern from the theory of quasi-Hamiltonian actions; in the Abelian case it extends this theory, but it is totally different for non-Abelian groups.

Poisson Lie group momentum maps

All manifolds and Lie groups are finite-dimensional. The theory is due to Lu [1990], Lu and Weinstein [1990].

A Lie group G is a *Poisson Lie group* if it is simultaneously a Poisson manifold and group multiplication and inversion are Poisson maps. Let $\varpi_G \in \mathfrak{X}^2(G)$(= bivector fields) denote the Poisson tensor of G. Let (M, ϖ_M) be a Poisson manifold.

A *Poisson action* of the Poisson Lie group G on (M, ϖ_M) is a smooth (left) action $G \times M \to M$ which is, in addition, a Poisson map (with $G \times M$ endowed with the product Poisson structure $\varpi_G \times \varpi_M$. (Functions on G Poisson commute with functions on M).

The Poisson tensor ϖ_G of a Poisson Lie group G, with Lie algebra $\mathfrak{g}$, necessarily vanishes at the identity element $e \in G$, which then allows for the definition of the *intrinsic derivative* $\epsilon : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ by $\epsilon(A) := (\mathfrak{L}_X \varpi_G)_e$, where $X \in \mathfrak{X}(G)$ is an arbitrary vector field satisfying $X_e = A$ and $\mathfrak{L}_X$ denotes the Lie derivative in the direction X. The dual map $\epsilon^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies the Jacobi identity, thus endowing $\mathfrak{g}^*$ with a Lie algebra structure. The unique connected and simply connected Lie group G^* whose Lie algebra is $\mathfrak{g}^*$ is called the *dual group* of G. The Lie group G^* has a unique Poisson structure ϖ_{G^*} relative to which G^* is a Poisson Lie group such that the intrinsic derivative of ϖ_{G^*} is the Lie bracket on $\mathfrak{g}$. If G is connected and simply connected, the intrinsic derivative ϵ is a cocycle which uniquely determines both Poisson Lie tensors ϖ_G and ϖ_{G^*}.

Let $G \times M \to M$ be a left Poisson action of the Poisson Lie group (G, ϖ_G) on the Poisson manifold (M, ϖ_M). A smooth map $J : M \to G^*$, if it exists, is called a *momentum map* of this action if

$$A^* + \varpi_M \left(\cdot, J^* A^l\right) = 0, \quad \text{for all} \quad A \in \mathfrak{g}.$$

Here, A^* denotes the fundamental (or infinitesimal generator) vector field on M induced by the infinitesimal action of $A \in \mathfrak{g}$, i.e.,

$$A^*(m) := \left.\frac{d}{dt}\right|_{t=0} \exp(tA) \cdot m, \quad \text{for all} \quad m \in M,$$

where $g \cdot m$ denotes the action of $g \in G$ on $m \in M$. The second term in the definition is interpreted in the following way. Since $\mathfrak{g}$ is the dual of $\mathfrak{g}^*$ (which is the Lie algebra of G^*), we think of A as a linear map on $\mathfrak{g}^*$ and hence it defines a unique left invariant one-form $A^l \in \Omega^1(G^*)$ whose value at the identity is A, i.e., $(A^l)_a(v) = \left\langle A, \mathsf{L}_{a^{-1}} v \right\rangle$ for every $a \in G^*$ and $v \in \mathsf{T}_a G^*$, where $\mathsf{L}_{a^{-1}}$ denotes both the left translation by $a^{-1} \in G^*$ in G^* and its tangent map on $\mathsf{T}G^*$.

Assume now that the Poisson manifold (M, ϖ_M) is symplectic with symplectic form ω and let us unwind the definition in this case. For any $X_m \in \mathsf{T}_m M$ we have

$$\omega_m(A^*_m, X_m) = (\varpi_M)_m \left(\left(\varpi^\sharp_M \right)^{-1} A^*_m, \left(\varpi^\sharp_M \right)^{-1} X_m \right) = -\left(J^* A^l \right)_m X_m$$
$$= -\left(A^l \right)_{J(m)} (\mathsf{T}_m J(X_m)) = -\left\langle A, \mathsf{L}_{J(m)^{-1}} \mathsf{T}_m J(X_m) \right\rangle$$
$$= -\left\langle A, (\delta J)_m(X_m) \right\rangle,$$

where $\mathsf{T}_m J : \mathsf{T}_m M \to \mathsf{T}_{J(m)} G^*$ is the derivative (tangent map) of $J : M \to G^*$ and $\delta J \in \Omega^1(M, \mathfrak{g}^*)$, defined by the last equality, is its *left logarithmic derivative*.

Key observation: The identity $\omega_m(A^*_m, X_m) + \langle A, (\delta J)_m(X_m) \rangle = 0$ proved above does not use the Poisson Lie group structure on G. This identity makes sense if the momentum map is replaced by a smooth map $J : M \to H$ with values in an arbitrary Lie group H, as long as there is a duality between the Lie algebras of G and H. This observation leads to our generalization of Lu's momentum map. To define this generalization, we need a few preliminary concepts, inspired by their counterparts in the theory of Poisson Lie groups.

Dual pairs of Lie algebras

A *dual pair of Lie algebras* (not necessarily finite dimensional) consists of two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, which are in duality through a given (weakly) non-degenerate bilinear map $\kappa : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$. Like in functional analysis, we often write the dual pair as $\kappa(\mathfrak{g}, \mathfrak{h})$. Intuitively, we think of $\mathfrak{h}$ as the dual vector space of $\mathfrak{g}$, endowed with its own Lie bracket operation, so sometimes we write $\mathfrak{g}^* := \mathfrak{h}$, even though $\mathfrak{g}^*$ is not necessarily the functional analytic dual of $\mathfrak{g}$.

Two Lie groups G and H are said to be *dual* to each other if there exists a bilinear form $\kappa : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$ relative to which the associated Lie algebras are in duality. We use the notation $\kappa(G, H)$ in this case. As for Lie algebras, we often write $G^* := H$, intuitively thinking of G^* as the dual group, as in the theory of Poisson Lie groups.

The notion of a dual pair of Lie algebras involves only the underlying vector spaces, while the Lie brackets play no role. We introduce a more rigid concept of duality, which takes all structures into account. For a given dual pair $\kappa(\mathfrak{g},\mathfrak{h})$ of Lie algebras, define a bilinear skew-symmetric bracket on the double $\mathfrak{d} = \mathfrak{g} \times \mathfrak{h}$ by

$$[(A,\mu),(B,\nu)] = ([A,B]_{\mathfrak{g}} - \mathrm{ad}^*_{\mu} B + \mathrm{ad}^*_{\nu} A, [\mu,\nu]_{\mathfrak{h}} - \mathrm{ad}^*_{A} \nu + \mathrm{ad}^*_{B} \mu),$$

for $A, B \in \mathfrak{g}$, $\mu,\nu \in \mathfrak{h}$, where the infinitesimal coadjoint actions are defined with respect to κ by

$$\kappa(B, \mathrm{ad}^*_{A} \mu) = \kappa([A,B]_{\mathfrak{g}}, \mu),$$
$$\kappa(\mathrm{ad}^*_{\mu} A, \nu) = \kappa(A, [\mu,\nu]_{\mathfrak{h}}).$$

However, this bracket does not satisfy the Jacobi identity, in general. A dual pair $\kappa(\mathfrak{g},\mathfrak{h})$ of Lie algebras is called a *Lie bialgebra*, if this bracket on $\mathfrak{d} = \mathfrak{g} \times \mathfrak{h}$ is a Lie bracket. In this case, we denote the double by $\mathfrak{g} \bowtie \mathfrak{h}$.

Examples 1.) $\kappa(\mathfrak{g},\mathfrak{h})$ dual pair with $\mathfrak{h}$ Abelian, so the coadjoint action $\mathrm{ad}^*_\mu : \mathfrak{g} \to \mathfrak{g}$ is trivial for every $\mu \in \mathfrak{h}$. Hence the bracket on the double $\mathfrak{d}$ simplifies to $[(A,\mu),(B,\nu)] = ([A,B]_{\mathfrak{g}}, -\mathrm{ad}^*_A \nu + \mathrm{ad}^*_B \mu)$, so $\mathfrak{d}$ is the semidirect product $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{h}$ of Lie algebras, where $\mathfrak{g}$ acts on $\mathfrak{h}$ by the κ-coadjoint action. The Jacobi identity always holds and thus $\kappa(\mathfrak{g},\mathfrak{h})$ is a Lie bialgebra.

2.) (Group of volume-preserving diffeomorphisms) M be a compact manifold with a volume form μ. Then the group $G = \mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms is a Fréchet Lie group (already known to Hamilton [1982]). Its Lie algebra is $\mathfrak{g} = \mathfrak{X}_\mu(M) := \{X \in \mathfrak{X}(M) \mid \mathfrak{L}_X \mu = 0\}$. Hence, we also identify $\mathfrak{X}_\mu(M)$ with $\Omega^{\dim M-1}_{\mathrm{cl}}(M)$ via $X \mapsto \mathrm{i}_X\mu$, where $\Omega^k_{\mathrm{cl}}(M)$ denotes the space of closed k-forms on M. Thus $\Omega^1(M)/\mathrm{d}\Omega^0(M)$ is the regular dual with respect to the weakly non-degenerate integration paring

$$\kappa(X,\alpha) := (-1)^{\dim M-1} \int_M (\mathrm{i}_X\mu) \wedge \alpha = \int_M (\mathrm{i}_X\alpha)\,\mu.$$

We now observe that a 1-form α can be interpreted as a trivial principal circle bundle with curvature $\mathsf{d}\alpha$. From this point of view, $\Omega^1(M)/\mathsf{d}\Omega^0(M)$ parametrizes equivalence classes of connections on a trivial principal circle bundle. Thus, it is natural to think of it as the Lie algebra of the Abelian group $H := \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ of all principal circle bundles with connections, modulo gauge equivalence. This heuristic argument can be made rigorous using the theory of Cheeger-Simons differential characters. So we get a dual pair $\kappa(\mathsf{Diff}_\mu(M), \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)))$ of Lie groups. For later use, it is convenient to introduce the notation $\widehat{\mathfrak{h}}^2(M, \mathsf{U}(1))$ for the Lie algebra $\Omega^1(M)/\mathsf{d}\Omega^0(M)$ of $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$.

3.) (Group of symplectomorphisms) (M, ω) compact symplectic manifold. The group $G = \mathsf{Diff}_\omega(M)$ of symplectomorphisms is a Fréchet Lie group (Kriegl and Michor [1997]) with Lie algebra

$$\mathfrak{g} = \mathfrak{X}_\omega(M) := \{X \in \mathfrak{X}(M) \mid \mathsf{d}\mathsf{i}_X\omega = 0\} \ni X \stackrel{\sim}{\longleftrightarrow} \mathsf{i}_X\omega \in \Omega^1_{\mathsf{cl}}(M).$$

Thus the regular dual with respect to the natural integration paring

$$\kappa(X,\alpha) := \frac{(-1)^{\dim M-1}}{\left(\frac{1}{2}\dim M - 1\right)!}\int_M (\mathrm{i}_X\omega)\wedge\alpha$$

is $\widehat{\mathfrak{h}}^{\dim M}(M,\mathsf{U}(1)) := \Omega^{\dim M-1}(M)/\mathrm{d}\Omega^{\dim M-2}(M)$. The prefactor in front of the integral turns out to be a convenient choice in later computations. As in the case of volume-preserving diffeomorphisms, the Abelian Lie algebra $\widehat{\mathfrak{h}}^{\dim M}(M,\mathsf{U}(1))$ is integrated by the group $\widehat{\mathsf{H}}^{\dim M}(M,\mathsf{U}(1))$ of Cheeger-Simons differential characters with degree $\dim M$. These can be thought of as equivalence classes of circle n-bundles with connections in the sense of higher differential geometry.

Remark: Both $\mathrm{Diff}_\mu(M)$, $\mathrm{Diff}_\omega(M)$ have an Abelian dual group. Hence they are special cases of the first example; thus both dual pairs $(\mathfrak{X}_\mu(M),\widehat{\mathfrak{h}}^2(M,\mathsf{U}(1)))$ and $(\mathfrak{X}_\omega(M),\widehat{\mathfrak{h}}^{\dim M}(M,\mathsf{U}(1)))$ are Lie bialgebras.

Ignoring the particularities of the infinite-dimensional setting for a moment, Drinfeld's theorem states that there are essentially unique connected and simply connected Poisson Lie groups, whose Lie algebras are $\mathfrak{X}_\mu(M)$ and $\mathfrak{X}_\omega(M)$, respectively. We do not know if the groups $\mathsf{Diff}_\mu(M)$ or $\mathsf{Diff}_\omega(M)$ carry a non-trivial Poisson Lie structure integrating the above Lie bialgebras (this would require to find a non-trivial integration of the adjoint action). Moreover, we are not aware of *any* Poisson Lie structure on these groups, such that the actions used later in the examples are Poisson maps.

4.) (Gauge group) $P \to M$ right principal G-bundle, M compact connected boundaryless. The group $\mathsf{Gau}(P)$ of gauge transformation is identified with the space of sections of $P \times_G G := (P \times G)/G$ and thus is a Fréchet Lie group with Lie algebra $\mathfrak{gau}(P) = \Gamma^\infty(\mathrm{Ad}\, P)$, the space of sections of the adjoint bundle $\mathrm{Ad}\, P := (P \times \mathfrak{g})/G$. Denote the dual of the adjoint bundle by $\mathrm{Ad}^*\, P := (P \times \mathfrak{g}^*)/G$, the action of G on $\mathfrak{g}^*$ being the left coadjoint action.

The natural pairing

$$\kappa(\phi,\alpha) = \int_M \langle \phi, \alpha \rangle\,, \qquad \phi \in \mathfrak{gau}(P), \qquad \alpha \in \Omega^{\dim M}(M, \mathrm{Ad}^* P),$$

identifies $\Omega^{\dim M}(M, \mathrm{Ad}^* P)$ as the regular dual to $\mathfrak{gau}(P)$. In particular, if M is endowed with a volume form μ, then $\mathfrak{gau}^*(P) = \Gamma^\infty(\mathrm{Ad}^* P)$ is the dual by integration against μ:

$$\langle \cdot, \cdot \rangle_{\mathrm{Ad}} : \Gamma^\infty(\mathrm{Ad}\, P) \times \Gamma^\infty(\mathrm{Ad}^* P) \to \mathbb{R}, \qquad (\rho, \varrho) \mapsto \int_M \langle \rho, \varrho \rangle\, \mu.$$

Moreover, an Ad_G-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ identifies $\mathfrak{gau}^*(P)$ with $\mathfrak{gau}(P)$ so that $\mathfrak{gau}(P)$ is self-dual in this case.

Group-valued momentum maps

Let M be G-manifold endowed with a symplectic form ω (not necessarily G-invariant). A *group-valued momentum map* is a pair (J, κ), where $\kappa(G, G^*)$ is a dual pair of Lie groups and $J : M \to G^*$ is a smooth map satisfying

$$\mathsf{i}_{A^*}\omega + \kappa(A, \delta J) = 0, \quad \forall A \in \mathfrak{g};$$

A^* is the fundamental vector field on M induced by $A \in \mathfrak{g}$, $\delta J \in \Omega^1(M, \mathfrak{g}^*)$ is the left logarithmic derivative of J, $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{g}^*$ is the Lie algebra of G^*.

Examples: 1.) (Standard momentum map) View $G^* = \mathfrak{g}^*$ as an Abelian group, $\kappa : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ duality pairing. Thus a $\mathfrak{g}^*$-valued momentum map is $J : M \to \mathfrak{g}^*$ satisfying the usual relation

$$\mathsf{i}_{A^*}\omega + \mathsf{d}J_A = 0, \quad \forall A \in \mathfrak{g}.$$

Here $J_A = \kappa(A, J) : M \to \mathbb{R}$ and the Abelian character of $\mathfrak{g}^*$ implies $\mathsf{d}J_A = \kappa(A, \mathsf{T}J) = \kappa(A, \delta J)$.

2.) (Lie algebra valued momentum map) If $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a continuous, weakly non-degenerate, Ad_G-invariant symmetric bilinear form, one identifies formally the functional analytic dual $\mathfrak{g}^*$ with $\mathfrak{g}$. So, a Lie algebra-valued momentum map is a smooth map $J : M \to \mathfrak{g}$ such that, for all $A \in \mathfrak{g}$, the component functions $J_A : M \to \mathbb{R}$ defined by $J_A(m) = \kappa(J(m), A)$, $m \in M$, satisfy

$$\mathrm{i}_{A^*}\omega + \mathrm{d}J_A = 0.$$

It is clear that such a Lie algebra-valued momentum map can be regarded as a group-valued momentum map with respect to the dual pair $\kappa(G, \mathfrak{g})$ where $\mathfrak{g}$ is viewed as an Abelian Lie group.

3.) (Poisson momentum map) (M, ω) finite-dimensional symplectic. Suppose a Poisson Lie group G acts on M in a Poisson manner. G^* is the dual group. $\kappa : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ duality pairing. Hence a G^*-valued momentum map is a smooth map $J : M \to G^*$ satisfying

$$\mathrm{i}_{A^*}\omega + \kappa(A, \delta J) = 0.$$

As we have explained above, this is just a reformulation of the usual Lu momentum map relation in the context of symplectic geometry.

In other words, our group-valued momentum map is the natural generalization of the Poisson momentum map if the Lie group G is not necessarily a Poisson Lie group.

4.) (Circle-valued momentum map) (M, ω) symplectic manifold, symplectic $G = \mathsf{U}(1)$-action. We let $G^* = \mathsf{U}(1)$ and take $\kappa : \mathfrak{u}(1) \times \mathfrak{u}(1) \to \mathbb{R}$, $\kappa(x, y) = xy$ (usual multiplication of real numbers under the identification $\mathfrak{u}(1) = \mathbb{R}$), as the pairing between the Lie algebras of G and G^*. Thus, a map $J : M \to \mathsf{U}(1)$ is a group-valued momentum map if and only if

$$\mathbf{i}_{1^*}\omega + \delta J = 0,$$

i.e., recover the usual definition of a circle-valued momentum map.

5.) (Symplectic torus) (V, ω) symplectic vector space. A lattice Λ in V is a discrete subgroup of the additive group $(V, +)$. Thus Λ acts naturally on V by translations. The symplectic structure is invariant under this action and hence descends to a symplectic form ω_T on the torus $T = V/\Lambda$.

Since the translation action of V commutes with the lattice action, the additive group $(V, +)$ also acts symplectically on the torus. The action of V on itself has the momentum map

$$J : V \to V^*, \qquad v \mapsto \omega(v, \cdot).$$

However, J is not invariant under the lattice action and so does *not* descend to a momentum map for the induced action of V on the torus T. Indeed, it is well-known that, for cohomological reasons, the symplectic action on the torus does not admit a standard momentum map. Rather, J transforms as

$$J(v + \lambda) = J(v) + \omega(\lambda, \cdot), \quad \lambda \in \Lambda.$$

Thus if $\omega(\lambda_1, \lambda_2) \in \mathbb{Z}$ holds for all $\lambda_1, \lambda_2 \in \Lambda$, then J is equivariant with respect to the dual lattice action

$$\Lambda^* = \{\alpha \in V^* \mid \alpha(\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

In this case, J induces a V^*/Λ^*-valued momentum map J_T on the torus T. It is interesting to note that the integrality condition $\omega(\lambda_1, \lambda_2) \in \mathbb{Z}$ is equivalent to ω_T being prequantizable.

6.) (Cylinder-valued momentum map) In a finite-dimensional context, Condevaux-Dazord-Molino [1988] introduced a momentum map with values in the cylinder $C := \mathfrak{g}^*/H$, where H is the holonomy group of a flat connection on some bundle constructed in terms of the symplectic form and the action (in our language, α defined in the next section plays the role of the connection form). If the holonomy group H is discrete, then C is a Lie group with Lie algebra $\mathfrak{g}^*$. Thus C is a dual group. Under the identification of the Lie algebra $\mathfrak{c} = \mathfrak{g}^*$, the cylinder-valued momentum map satisfies the defining identity for a group valued momentum map (shown in Ortega-Ratiu [2003], Theorem 5.2.8), and hence is a group-valued momentum map. The group-valued momentum map for the symplectic torus discussed in the previous example is also the cylinder-valued momentum map (shown in Example 5.2.5 of Ortega-Ratiu [2003].

The case when the holonomy group $H \subseteq \mathfrak{g}^*$ has accumulation points is pathological both in the framework of cylinder- as well as group-valued momentum maps; more on this later.

Despite its general nature, a group-valued momentum map still captures conserved quantities of the dynamical system, i.e., it has the *Noether property*.

Let (M, ω) be symplectic G-manifold. Suppose that the action has a G^*-valued momentum map $J : M \to G^*$. Let $h \in C^\infty(M)$ for which the Hamiltonian vector field X_h exists and has a local flow. (Recall that vector fields on Fréchet manifolds do not need to have flows. This is more or less equivalent to local in time solutions of the corresponding partial differential equation.) If h is G-invariant, then J is constant along the integral curves of X_h.

Existence and uniqueness

Define a $\mathfrak{g}^*$-valued 1-form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ by $\mathrm{i}_{A^*}\omega + \kappa(A, \alpha) = 0$, i.e.,

$$\kappa(A, \alpha_m(X_m)) = \omega_m(X_m, A^*_m) \quad \text{for all } X_m \in \mathsf{T}_m M,\ A \in \mathfrak{g}.$$

In infinite dimensions, the dual pairing κ is mostly weakly non-degenerate. In such cases, there might not exists $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ satisfying this identity, although α is unique, if it exists. We will assume from now on that we have such an α.

The definition of the group-valued momentum map implies that the G-action on M admits a G^*-valued momentum map if and only if $\alpha = \delta J$ for some smooth function $J : M \to G^*$, i.e., $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ is log-exact.

Below, $[\alpha \wedge \beta]$ means the wedge product of the $\mathfrak{g}^*$-valued forms α and β on M associated to the bracket operation on $\mathfrak{g}^*$ (as the Lie algebra of G^*).

(i) $\alpha \in \Omega^1(M, \mathfrak{g}^*)$ on M defined above satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$$

if and only if $\mathfrak{L}_{A^*}\omega = \frac{1}{2}\kappa(A, [\alpha \wedge \alpha])$ holds for all $A \in \mathfrak{g}$. In this case, we say that the G-action on (M, ω) is *G^*-symplectic*. If G^* is Abelian, then the notions of symplectic and G^*-symplectic group actions coincide.
(ii) If the G-action on M admits a G^*-valued momentum map $J : M \to G^*$, then $\delta J \in \Omega^1(M, \mathfrak{g}^*)$ satisfies the Maurer-Cartan equation.

Strengthen this statement in terms of the period map. Need the notion of regular Lie group.

A Lie group G modeled on a locally convex space is *regular* if for each curve $c \in C^\infty([0,1],\mathfrak{g})$, the initial value problem $\delta\eta(t) := \mathsf{L}_{\eta(t)^{-1}}\dot\eta(t) = c(t)$, $\eta(0) = e$, has a solution $\eta_c \in C^\infty([0,1],G)$ and the endpoint evaluation map $C^\infty([0,1],\mathfrak{g}) \ni c \mapsto \eta_c(1) \in G$ is smooth.
G is regular $\Rightarrow$ it has a smooth exponential function. All Banach (so, in particular, all finite dimensional) Lie groups are regular.

Fix a point $m_0 \in M$ and consider a piece-wise smooth loop $\gamma : I \to M$ based at m_0. Pulling back $\alpha \in \Omega^1(M,\mathfrak{g}^*)$ by γ yields a $\mathfrak{g}^*$-valued 1-form $\gamma^*\alpha$ on the interval I. Denote by $\eta_\gamma \in C^\infty(I,G^*)$ the solution of the initial value problem

$$\delta\eta = \gamma^*\alpha, \qquad \eta(0) = e,$$

which exists if G^* is regular. Evaluating η_γ at the endpoint 1, we obtain the *period homomorphism*

$$\mathrm{per}_\alpha : \pi_1(M,m_0) \ni [\gamma] \mapsto \eta_\gamma(1) \in G^*,$$

where $[\gamma]$ is the homotopy class of the loop γ.

Let (M, ω) be a connected symplectic manifold and $\kappa(G, G^*)$ a dual pair of Lie groups. In infinite dimensions, we additionally assume that G^* is a regular Lie group. Suppose that the the framed formula has a solution $\alpha \in \Omega^1(M, \mathfrak{g}^*)$. Let G act on M in a G^*-symplectic way (i.e., α satisfies the Maurer-Cartan equation $\mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \iff \mathfrak{L}_{A^*}\omega = \frac{1}{2}\kappa(A, [\alpha \wedge \alpha]), \forall A \in \mathfrak{g}$). Then there exists a G^*-valued momentum map if and only if the period homomorphism $\mathrm{per}_\alpha : \pi_1(M, m_0) \to G^*$ is trivial. Moreover, the momentum map is unique up to translation by a constant element $h \in G^*$.

Example of a symplectic Lie group action without group valued momentum map: Let $M = \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^4 \ni (\varphi_1, \varphi_2, \psi_1, \psi_2)$. Endow M with the symplectic form

$$\omega = \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 + \sqrt{2}\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2.$$

The circle action given by $\lambda \cdot (\varphi_1, \varphi_2, \psi_1, \psi_2) = (\varphi_1 - \lambda, \varphi_2, \psi_1 - \lambda, \psi_2)$ is clearly symplectic. The framed equation has the solution

$$\alpha = \mathrm{d}\varphi_2 + \sqrt{2}\mathrm{d}\psi_2.$$

As the generators of $\pi_1(M)$ we take the four natural loops γ_i, where, for $1 \leq i \leq 4$, the loop $\gamma_i : I \to M$ winds once around the i-th circle in $M = (\mathbb{R}/\mathbb{Z})^4$. The pull-back of α by γ_1 and γ_3 vanishes and we find

$$\gamma_2^*\alpha = \mathrm{d}t \quad \text{and} \quad \gamma_4^*\alpha = \sqrt{2}\mathrm{d}t,$$

where $t \in I = [0,1]$. Since there are only two connected one-dimensional Lie groups, the only possible choices for the dual group are $G^* = \mathbb{R}$ and $G^* = \mathbb{R}/\mathbb{Z}$. In both cases, the initial value problem

$$\delta\eta = \gamma^*\alpha, \qquad \eta(0) = e$$

has the unique solutions $\eta_{\gamma_2}(t) = t$ and $\eta_{\gamma_4}(t) = \sqrt{2}t$. Thus neither for $G^* = \mathbb{R}$ nor for $G^* = \mathbb{R}/\mathbb{Z}$ the period homomorphism is trivial and hence no group-valued momentum map exists for this action.

This example exhibits another phenomenon that is particular for group-valued momentum maps: the action of a subgroup may not posses a group-valued momentum map even if the bigger group has a group-valued momentum map. In fact, the action $(\lambda_1, \lambda_2) \cdot (\varphi_1, \varphi_2, \psi_1, \psi_2) = (\varphi_1 - \lambda_1, \varphi_2, \psi_1 - \lambda_2, \psi_2)$ by $G = S^1 \times S^1$ has a group-valued momentum map (which is the product of two copies of the one discussed in example 6, the symplectic torus) but the action of the diagonally embedded circle has no group-valued momentum map as we have just seen.

Equivariance and Poisson property

$\kappa(G, G^*)$ dual pair of Lie groups. G acts on (M, ω) with group-valued momentum map $J : M \to G^*$. In which sense is J equivariant? We assume, for simplicity, that $(G^*, +)$ is Abelian, identity element $0 \in G^*$. In our infinite dimensional examples this is always the case.

A left action $\Upsilon : G \times G^* \to G^*$ is called a *coconjugation action* if it integrates the coadjoint action, that is, $\delta_\eta \Upsilon_g(\eta \cdot \mu) = \mathrm{Ad}^*_{g^{-1}} \mu$, $\forall g \in G$, $\eta \in G^*$, and $\mu \in \mathfrak{g}^*$, where

- Ad^* is defined with respect to the duality pairing κ by

$$\kappa(A, \mathrm{Ad}^*_g \mu) = \kappa(\mathrm{Ad}_g A, \mu),$$

- $\eta \cdot \mu := \mathrm{T}_e \mathrm{L}_\eta(\mu) \in \mathrm{T}_\eta G^*$,
- $\delta_\eta \Upsilon_g(\eta \cdot \mu) \in \mathfrak{g}^*$ denotes the (left) logarithmic derivative at η of the map $\Upsilon_g : G^* \to G^*$ in the direction $\eta \cdot \mu \in \mathrm{T}_\eta G^*$.

If, moreover, $\Upsilon_g(\zeta + \eta) = \Upsilon_g(\zeta) + \Upsilon_g(\eta)$ holds for all $\zeta, \eta \in G^*$, then we say that the coconjugation action is *standard*.

A coconjugation does not always exist. Even it is exists, it does not have to be unique; nonetheless, the class of coconjugation actions is rather rigid.

Let $\Upsilon : G \times G^* \to G^*$ be a standard coconjugation action.
1.) If $\tilde{\Upsilon} : G \times G^* \to G^*$ is another coconjugation action (not necessarily standard), then there exists a map $c : G \times \pi_0(G^*) \to G^*$ such that $\tilde{\Upsilon} = \Upsilon + c$.
2.) Conversely, a map $c : G \to G^*$ defines a coconjugation action $\tilde{\Upsilon} := \Upsilon + c$ if and only if c is a 1-cocycle with respect to Υ, i.e., it satisfies $c(gh) = c(g) + \Upsilon_g(c(h))$ for all $g, h \in G$.

Example: The coadjoint representation is a standard coconjugation action of G on $G^* := \mathfrak{g}^*$. Since $\mathfrak{g}^*$ is connected, the previous proposition establishes a bijection between coconjugation actions on $\mathfrak{g}^*$ and 1-cocycles $c : G \to \mathfrak{g}^*$. The coconjugation action corresponding to a 1-cocycle c is the affine action $(g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}} \mu + c(g)$, which plays an important role for classical non-equivariant momentum maps (the Souriau cocycle).

In the classical setting $G^* \equiv \mathfrak{g}^*$ The derivative of a 1-cocycle $c : G \to \mathfrak{g}^*$ yields a 2-cocycle on the Lie algebra which in turn uniquely defines an affine Poisson structure on $\mathfrak{g}^*$. We will show that this extends to the general case and every coconjugation action gives rise to a Poisson structure on G^*.

Poisson structures in infinite dimensions need to be treated with caution! In our case, we can use the group structure to circumvent most of the technical issues. In finite dimensions, there are many equivalent ways to look at a bivector field on G^*. In a left-trivialization $\mathsf{T}G^* \simeq G^* \times \mathfrak{g}^*$, a bivector field π_{G^*} is a smooth map $G^* \to \bigwedge^2 \mathfrak{g}^*$. Using reflexivity $\mathfrak{g}^{**} = \mathfrak{g}$, this may, equivalently, be viewed as a map

$$\pi_{G^*} : G^* \times \mathfrak{g} \to \mathfrak{g}^*.$$

It is this latter form that we adopt as the definition of a bivector field on G^* in the infinite-dimensional setting.

Recall that for a smooth map $f : M \to G$ the *left logarithmic derivative* is the Lie algebra-valued 1-form δf on M defined by $\delta_m f(X) = f(m)^{-1} \cdot \mathsf{T}_m f(X)$, where $X \in \mathsf{T}_m M$.

Every standard coconjugation action $\Upsilon : G \times G^* \to G^*$ defines a Poisson Lie structure $\pi_{G^*} : G^* \times \mathfrak{g} \to \mathfrak{g}^*$ on G^* by

$$\pi_{G^*}(\eta, A) = \delta_{(e,\eta)} \Upsilon(A, 0) = -\eta \cdot \mathsf{T}_e \Upsilon_\eta(A).$$

Moreover, if $\tilde{\Upsilon}$ is another coconjugation (not necessarily standard) and the map $c : G \times \pi_0(G^*) \to G^*$ from from the previous proposition satisfies $c_{\zeta+\eta} = c_\zeta + c_\eta - c_0$ for all $\zeta, \eta \in G^*$, then the associated bivector field $\tilde{\pi}_{G^*}$ is an affine Poisson structure.

By construction, Υ and the Poisson tensor π_{G^*} are connected by

$$\eta \cdot \pi_{G^*}(\eta, A) = \mathsf{T}_e \Upsilon_\eta(A) \in \mathsf{T}_\eta G^*, \qquad \forall \eta \in G^*,\ \forall A \in \mathfrak{g}.$$

In Poisson geometry, actions satisfying this relation with respect to a Poisson structure on a Lie group are called *dressing actions*.

Example: Dual pair of Lie groups $\kappa(\mathsf{Diff}_\mu(M), \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)))$. The coadjoint action of a diffeomorphism is given by pull-back. Thus a natural coconjugation is given by

$$\mathsf{Diff}_\mu(M) \times \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)), \quad (\phi, h) \mapsto (\phi^{-1})^* h.$$

The induced Poisson Lie structure on $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ is defined by

$$\widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \times \mathfrak{X}_\mu(M) \to \Omega^1(M)/\mathsf{d}\Omega^0(M), \quad (h, X) \mapsto [\mathsf{i}_X \mathsf{curv}_h].$$

Under the integration pairing, this corresponds to

$$\widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \times \mathfrak{X}_\mu(M) \times \mathfrak{X}_\mu(M) \to \mathbb{R} \quad (h, X, Y) \mapsto \int_M \mathsf{curv}_h(X, Y)\, \mu.$$

For fixed $h \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$, this is precisely the Lichnerowicz cocycle on $\mathfrak{X}_\mu(M)$ defined by the 2-form curv_h. In other words, the Lichnerowicz cocycle is derived from the pull-back action. $\Diamond$

In the following, we also need the concept of the *left derivative* for maps whose domain is a Lie group, i.e., $F : G \to N$. The *left derivative of F at $g \in G$ in the direction $A \in \mathfrak{g}$* is defined by

$$\mathsf{T}^{\mathsf{L}}_g F(A) := \mathsf{T}_g F(g \cdot A); \quad \text{thus} \quad \mathsf{T}^{\mathsf{L}}_g F : \mathfrak{g} \to \mathsf{T}_{F(g)} N \quad \text{linear.}$$

So, if $f : G \to \mathbb{R}$, then $\mathsf{T}^{\mathsf{L}}_g f : \mathfrak{g} \to \mathbb{R}$ is linear, i.e., i.e., $\mathsf{T}^{\mathsf{L}}_g f \in \mathfrak{g}^*$.

Given coconjugation action, we say that the group-valued momentum map is *equivariant* if it is G-equivariant as a map $J : M \to G^*$. For classical momentum maps, there is a well-known equivalence between being equivariant and being a Poisson map. This is true for our group-valued momentum map.

For a smooth function $f : G^* \to \mathbb{R}$, the left derivative $\mathsf{T}^{\mathsf{L}}_{\eta} f : \mathfrak{g}^* \to \mathbb{R}$ at $\eta \in G^*$ is an element of the double dual $\mathfrak{g}^{**}$.

Need to reformulate the Poisson property without using $\mathfrak{g}^{**} = \mathfrak{g}$.

In finite dimensions, $\mathsf{T}^{\mathsf{L}}_{\eta} f \in \mathfrak{g}$, so if $f, g \in C^{\infty}(G^*)$,

$$\{f, g\}_{G^*}(\eta) = \kappa\left(\pi_{G^*}(\eta, \mathsf{T}^{\mathsf{L}}_{\eta} f), \mathsf{T}^{\mathsf{L}}_{\eta} g\right).$$

For $J : M \to G^*$, we calculate for any $X \in \mathsf{T}_m M$,

$$\begin{aligned} \mathsf{d}(f \circ J)(m)(X) &= \mathsf{d}f(J(m))(J(m) \cdot \delta_m J(X)) = \kappa(\mathsf{T}^{\mathsf{L}}_{J(m)} f, \delta_m J(X)) \\ &= -\omega_m((\mathsf{T}^{\mathsf{L}}_{J(m)} f) \cdot m, X). \end{aligned}$$

by the definition of the group-valeued momentum map.

Applying the definition of the momentum map again

$$\begin{aligned}\{f\circ J, g\circ J\}_M(m) &= \varpi_m(\mathsf{d}(f\circ J)(m), \mathsf{d}(g\circ J))(m)\\ &= \omega_m((\mathsf{T}^{\mathsf{L}}_{J(m)}f)\cdot m, (\mathsf{T}^{\mathsf{L}}_{J(m)}g)\cdot m).\end{aligned}$$

So J is Poisson map, i.e., $\{f\circ J, g\circ J\}_M = \{f,g\}_{G^*}\circ J$, if and only if

$$\omega_m(A\cdot m, B\cdot m) = \kappa(\pi_{G^*}(J(m),A),B), \quad \forall A,B\in\mathfrak{g}, \quad \forall m\in M.$$

This equation no longer relies on reflexivity to make sense and so we adopt it as the definition for J to be a Poisson map. The left-hand side defines the so called *non-equivariance* cocycle $\sigma_m(A,B) = \omega_m(A\cdot m, B\cdot m)$. Thus J is a Poisson map if and only if the Lie algebra cocycles σ_m and $\kappa(\pi_{G^*}(J(m),\cdot),\cdot)$ coincide.

If $J: M\to G^*$ is equivariant with respect to a given coconjugation action then it is Poisson relative to the induced Poisson tensor π_{G^*}.

Proof: $J: M\to G^*$ equivariant $\Rightarrow J(g\cdot m) = \Upsilon_g J(m)$ (Υ coconjugation) Thus, $\delta_m J(A\cdot m) = \pi_{G^*}(J(m),A)$ by the definition of π_{G^*}. On the other hand, $\kappa(B,\delta_m J(A\cdot m)) = \omega_m(A\cdot m, B\cdot m)$ and the claim follows. □

Momentum maps for group extensions

Short exact sequence of Lie groups

$$e \longrightarrow H \stackrel{\iota}{\longrightarrow} K \stackrel{\pi}{\longrightarrow} G \longrightarrow e.$$

Suppose K acts symplectically on (M, ω). Seek an expression of the momentum map for the K-action in terms of the momentum maps for the groups H and G, assuming they exist. Similar questions occur in the context of symplectic reduction by stages.

The induced short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{h} \stackrel{\iota}{\longrightarrow} \mathfrak{k} \stackrel{\pi}{\longrightarrow} \mathfrak{g} \longrightarrow 0$$

always splits as vector spaces but not necessarily as Lie algebras. Fix a splitting $\sigma : \mathfrak{g} \to \mathfrak{k}$ in the category of locally convex vector spaces and write $\mathfrak{h} \oplus_\sigma \mathfrak{g} = \mathfrak{k}$ for the corresponding direct sum. Thus, every $A \in \mathfrak{k}$ can be uniquely written as the sum $A = \iota(A_H) + \sigma(A_G)$ with $A_H \in \mathfrak{h}$ and $A_G = \pi(A) \in \mathfrak{g}$. Assume that $\mathfrak{h}$ is self-dual with respect to a pairing $\langle \cdot, \cdot \rangle$ and $\kappa(G, G^*)$ is a dual pair of Lie groups.

In this setting, if the induced action of H on M has a standard momentum map $J_H : M \to \mathfrak{h}$ with respect to the pairing $\langle \mathfrak{h}, \mathfrak{h} \rangle$ and there exists a map $J_\sigma : M \to G^*$ satisfying

$$\mathrm{i}_{\sigma(A_G)^*}\omega + \kappa(A_G, \delta J_\sigma) = 0, \qquad A_G \in \mathfrak{g},$$

then $J_K = (J_H, J_\sigma) : M \to \mathfrak{h} \times G^*$ is a group-valued K-momentum map with respect to the pairing $(\mathfrak{h} \oplus_\sigma \mathfrak{g}, \mathfrak{h} \oplus \mathfrak{g}^*) = \langle \mathfrak{h}, \mathfrak{h} \rangle + \kappa(\mathfrak{g}, \mathfrak{g}^*)$.

The identity in the statement is, formally, the momentum map relation for G. However, we do not assume that σ is a splitting on the level of Lie algebras. Hence G, or its Lie algebra $\mathfrak{g}$, does not act on M via σ and $J_\sigma : M \to G^*$ *is not a momentum map*. However, if G happens to act on M through a different splitting $\chi : G \to K$ which is a group section of π, then J_σ is the momentum map up to some twisting by J_H.

In this setting, let $\chi : \mathfrak{g} \to \mathfrak{k}$ be a Lie algebra homomorphism splitting the exact Lie algebra sequence above. Hence there is an induced (infinitesimal) $\mathfrak{g}$-action on M. Define $\tau_{\sigma\chi} := \chi - \sigma : \mathfrak{g} \to \mathfrak{h}$. Its dual map with respect to the chosen pairings is denoted by $\tau^*_{\sigma\chi} : \mathfrak{h}^* \to \mathfrak{g}^*$. Assume that the dual group G^* of G is Abelian. Then

$$J_\chi : M \to G^*, \qquad m \mapsto J_\sigma(m) \cdot \exp(\tau^*_{\sigma\chi} J_H(m))$$

is a group-valued momentum map for the G-action on M. Moreover, J_χ does not depend on the splitting σ.

Determine the momentum map for the action of a subgroup.

G acts symplectically on (M, ω) with group-valued momentum map $J : M \to G^*$. Let $\iota : H \to G$ be a Lie group homomorphism; hence H acts through G on M. Fix a dual group H^* of H. Suppose that there is a Lie group homomorphism $\rho : G^* \to H^*$ whose associated Lie algebra homomorphism $\rho : \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual of $\iota : \mathfrak{h} \to \mathfrak{g}$ with respect to $\kappa(\mathfrak{g}, \mathfrak{g}^*)$ and $\langle \mathfrak{h}, \mathfrak{h}^* \rangle$. Then $J^H := \rho \circ J : M \to H^*$ is a group-valued momentum map for the induced H-action.

These assumptions on ρ are automatically satisfied in finite-dimensions if G is connected and $\pi_1(G, e) = 1$. In infinite dimensions, however, the adjoint $\rho : \mathfrak{g}^* \to \mathfrak{h}^*$ of the linear map ι does not need to exist, and even if it exists, it does not necessarily integrate to a Lie group homomorphism (for this, we would need some regularity assumptions on the dual group H^* (Neeb [2006], Theorem III.1.5).

GLOBAL ANALYSIS OF SYMPLECTIC FIBER BUNDLES

Symplectic form on the space of sections

• $\pi : P \to M$ finite-dimensional right principal G-bundle.
• $\underline{l} : G \times \underline{F} \to \underline{F}$ left G-manifold $\underline{F}$.
• Form the associated fiber bundle $F = P \times_G \underline{F} \to M$.
• $\underline{F}$ carries a G-invariant symplectic form $\underline{\omega} \in \Omega^2(\underline{F})$
• Then there exists a unique symplectic structure ω_m on each fiber F_m such that the map

$$\iota_p : \underline{F} \to F_m, \qquad \underline{f} \mapsto [p, \underline{f}]$$

is a symplectomorphism for all $p \in P_m$.
• The fiber bundle F with the induced fiberwise symplectic structure ω is called a *symplectic fiber bundle* and we denote it by (F, ω).

• $\widehat{\omega} \in \Omega^2(F)$ is an *extension* of the fiberwise symplectic structure if its restriction to each fiber F_m coincides with ω_m.

In the theory of symplectic fiber bundles one usually tries to construct extensions that are symplectic forms again (fat bundles, Weinstein [1980], where one gives also a symplectic form on M).
We do not require $\widehat{\omega}$ to be non-degenerate.

- An extension $\widehat{\omega}$ is called *compatible with a connection* on P if the vertical and horizontal subbundles of the associated connection on F are orthogonal with respect to $\widehat{\omega}$. Such extensions always exist.
- M compact connected boundaryless, μ volume form.
- Space of sections $\mathscr{F} = \Gamma^\infty(F)$ is a Fréchet manifold. $\mathsf{T}_\phi\mathscr{F}$ for $\phi \in \underline{F}$, consists of vertical vector fields along ϕ, i.e., sections of $\phi^*\mathsf{V}F$, where $\mathsf{V}F \subset \mathsf{T}F$ is the vector subbundle of vertical vectors.
- Evaluation map $\mathrm{ev} : M \times \mathscr{F} \to F$, projection $\mathrm{pr}_M : M \times \mathscr{F} \to M$.

Define a 2-form Ω on $\mathscr{F}$ by

$$\Omega = \mu \widehat{*} \omega \equiv \int_M \mathrm{pr}_M^* \mu \wedge \mathrm{ev}^* \widehat{\omega},$$

(hat product introduced by Vizman [2011], generalized to sections of fiber bundles). The form Ω depends only on the fiber symplectic form ω and not on the extension $\widehat{\omega}$, since only the vertical part of the tangent vector plays a role.

$\mathsf{d}\Omega = (-1)^{\mathsf{dim}\, M} \mu \widehat{*} \mathsf{d}_\pi \omega = 0$, where d_π is the vertical differential along the fibers. Thus Ω is closed. Ω is symplectic because,

$$\Omega_\phi(Y_1, Y_2) = \int_M \omega_\phi(Y_1, Y_2)\, \mu, \quad Y_1, Y_2 \in \mathsf{T}_\phi \mathscr{F} = \Gamma^\infty(\phi^* \mathsf{V} F),$$

and ω is fiberwise non-degenerate by assumption. Conclusion:

M compact connected boundaryless manifold endowed with volume form μ. The space of sections $\mathscr{F}$ of the symplectic fiber bundle $(F = P \times_G \underline{F}, \omega)$ has the natural symplectic structure $\Omega = \mu \widehat{*} \omega$.

Action of Aut(P)

Every bundle automorphism ψ of F induces a diffeomorphism $\check{\psi}$ on the base M. Canonical smooth left action $\Upsilon : \mathsf{Aut}(F) \times \mathscr{F} \to \mathscr{F}$ is

$$\psi \cdot \phi = \psi \circ \phi \circ \check{\psi}^{-1}.$$

Let

$$\mathsf{Aut}_\mu(F) = \{\psi \in \mathsf{Aut}(F) \mid \check{\psi} \in \mathsf{Diff}_\mu(M)\}$$
$$\mathsf{Aut}_{\mu,\omega}(F) = \{\psi \in \mathsf{Aut}(F) \mid \check{\psi} \in \mathsf{Diff}_\mu(M),\ \psi^*\omega = \omega\}$$

be the subgroups of automorphisms of F inducing volume-preserving diffeomorphisms on the base and, in the second case, also preserving the fiber symplectic structure. $\Upsilon^*_\psi(\mu \widehat{*} \omega) = (\check{\psi}^*\mu)\widehat{*}(\psi^*\omega)$ shows that that this action preserves the symplectic form $\Omega = \mu \widehat{*} \omega$.

Natural to seek a momentum map for both the $\mathsf{Aut}_\mu(F)$- and $\mathsf{Aut}_{\mu,\omega}(F)$-actions on $\mathscr{F}$.

Serious problems: Such a momentum map would take values in the *distributional* dual. Gay-Balmaz and Vizman [2012] show that the action of (fiberwise) Hamiltonian automorphisms admits a momentum map satisfying $\langle X_h, J(\phi)\rangle = \int_M (\phi^* h)\mu$ after fixing points in M to identify Hamiltonian vector fields with smooth functions. In general, the functional

$$T_\phi : C^\infty(F, \mathbb{R}) \ni h \mapsto \int_M (\phi^* h)\mu \in \mathbb{R}$$

is not regular. E.g., if $M = \{x\} \Rightarrow T_\phi =$ delta distribution at $\phi(x)$.

The distributional character of the momentum map for $\mathrm{Diff}_\omega(F)$ is not surprising. The momentum map is conceptually the push-forward of μ, but the push-forward of a differential form into a higher-dimensional manifold results in a distributional measure.

To apply the strategy as in the general theory used below, we would need a Lie group whose Lie algebra consists of distributions; or, even better, we would need "distributional line bundles", i.e., singular bundles whose curvature is a distribution. $\lozenge$

From now on work only with $F = P \times_G \underline{F}$. It has more structure so momentum maps can be computed.

- $\widehat{\varphi} : P \to P$ principal bundle automorphism induces $\widehat{\varphi}_F = \widehat{\varphi} \times_G \mathrm{id}_{\underline{F}} : F \to F$ fiber bundle automorphism.
- By definition, $\widehat{\varphi}_F \circ \iota_p = \iota_{\widehat{\varphi}(p)}\ \forall p \in P$, where $\iota_p : \underline{F} \ni \underline{f} \mapsto [p, \underline{f}] \in F$. This implies $\iota_p^* \widehat{\varphi}_F^* \omega_{\varphi(m)} = \iota_{\widehat{\varphi}(p)}^* \omega_{\varphi(m)} = \underline{\omega} \Longleftrightarrow \widehat{\varphi}_F^* \omega_{\varphi(m)} = \omega_m$.
- Thus $\widehat{\varphi}_F$ leaves the symplectic structure invariant and we get a group homomorphism $\mathrm{Aut}_\mu(P) \to \mathrm{Aut}_{\mu,\omega}(F)$. Therefore, we could seek a momentum map for the $\mathrm{Aut}_\mu(P)$-action on $\mathscr{F}$.
- Divide this problem into its vertical and horizontal parts using the fundamental theorem about momentum maps of group extensions.
- Assigning to a principal bundle automorphism $\psi : P \to P$ the induced diffeomorphism $\check{\psi} : M \to M$, yields a group homomorphism $\mathrm{Aut}_\mu(P) \to \mathrm{Diff}_\mu(M)$, whose kernel is the gauge group $\mathrm{Gau}(P)$.

Thus we obtain a short exact sequence of Lie groups

$$\mathsf{id} \longrightarrow \mathsf{Gau}(P) \longrightarrow \mathsf{Aut}_\mu(P) \longrightarrow \mathsf{Diff}_{\mu,P}(M) \longrightarrow \mathsf{id},$$

where $\mathsf{Diff}_{\mu,P}(M) \subseteq \mathsf{Diff}_\mu(M)$ is the range of the third arrow.

• So we are in the setting discussed in in the general theory for group extensions. The fundamental theorem states that the momentum map for $\mathsf{Aut}_\mu(P)$ is obtained from the momentum maps for the gauge group and the 'partial' momentum map for $\mathsf{Diff}_{\mu,P}(M)$.

• Before we determine these, we record the common starting point. Infinitesimal generator vector field on $\mathscr{F}$ induced by $Y \in \mathfrak{aut}_\mu(P)$ is

$$Y^*_\phi = Y \circ \phi - \mathsf{T}\phi(Y_M) \in \mathsf{T}_\phi\mathscr{F} = \Gamma^\infty(\phi^*\mathsf{V}F), \quad \phi \in \mathscr{F},$$

where Y_M is the induced vector field on M. Using the identity

$$\mathsf{i}_{Y^*}(\mu \widehat{*} \omega) = \left(\mathsf{i}_{Y_M}\mu\right) \widehat{*} \omega + (-1)^{\dim M} \mu \widehat{*} \left(\mathsf{i}_Y \omega\right),$$

we obtain

$$\mathsf{i}_{Y^*}\Omega = \left(\mathsf{i}_{Y_M}\mu\right) \widehat{*} \widehat{\omega} + (-1)^{\dim M} \mu \widehat{*} \left(\mathsf{i}_{Y_F} \widehat{\omega}\right), \qquad Y \in \mathfrak{aut}_\mu(P),$$

$Y_F \in \mathfrak{X}(F)$ induced by natural map $\mathfrak{aut}_\mu(P) \to \mathfrak{aut}_{\mu,\omega}(F)$. So $\mathsf{i}_{Y^*}\Omega$ is sum of vertical and horizontal terms. So get first momentum map for $\mathsf{Gau}(P)$ from momentum map for the diffeomorphism group.

Momentum map for the gauge group

The following theorem (with a long proof) holds:

Let $F = P \times_G \underline{F} \to M$ be a symplectic fiber bundle modeled on the Hamiltonian G-manifold $(\underline{F}, \underline{\omega}, \underline{J})$, where $P \to M$ is a right principal G-bundle and M is compact connected boundaryless endowed with a volume form μ. The momentum map for the symplectic action $\mathrm{Gau}(P) \times \mathscr{F} \to \mathscr{F}$ is given by

$$\mathscr{J}_{\mathrm{Gau}} : \mathscr{F} \to \mathfrak{gau}^*(P), \qquad \phi \mapsto J_* \phi,$$

where $\mathfrak{gau}^*(P) = \Gamma^\infty(\mathrm{Ad}^* P) \cong \Omega^{\dim M}(M, \mathrm{Ad}^* P)$ is dual to $\mathfrak{gau}(P) = \Gamma^\infty(\mathrm{Ad}\, P)$ by the integration pairing $\langle \cdot, \cdot \rangle_{\mathrm{Ad}}$. Moreover, the bundle map $J = \mathrm{id}_P \times_G \underline{J} : F = P \times_G \underline{F} \to \mathrm{Ad}^* P = P \times_G \mathfrak{g}^*$ induces $J_* : \mathscr{F} \to \mathfrak{gau}^*(P)$ given for all $h \in \Gamma^\infty(\mathrm{Ad}^* P)$ and $\phi \in \Gamma^\infty(\mathrm{Ad}\, P)$ by

$$\langle J_* h, \phi \rangle_{\mathrm{Ad}} = \int_M \langle (\mathrm{id}_P \times_G \underline{J})(h(m)), \phi(m) \rangle \, \mu(m).$$

Momentum map for diffeomorphism groups

In the computations below we use a dual pair $\kappa(\mathrm{Diff}_{\mu,P}(M), \mathscr{H})$ but we do not specify $\mathscr{H}$ for the general case. The situations interesting to us are treated explicitly later.

By the fundamental theorem about momentum maps for group extensions, it is enough to split the infinitesimal sequence

$$0 \to \mathfrak{gau}(P) \to \mathfrak{aut}_\mu(P) \to \mathfrak{X}_\mu(M) \to 0,$$

and subsequently determine the 'partial' momentum map. This sequence naturally splits (in the category of locally convex vector spaces, not as Lie algebras) using a principal connection in P. Denote by $\Gamma : \mathfrak{X}_\mu(M) \to \mathfrak{gau}_\mu(P)$ the induced vector space homomorphism. Recall that the curvature of the principal connection is the obstruction for the lift $\Gamma : \mathfrak{X}_\mu(M) \to \mathfrak{aut}_\mu(P)$ to be a Lie algebra homomorphism. By the fundamental theorem about momentum maps for group extensions, the 'partial' momentum map $J_\Gamma : \mathscr{F} \to \mathscr{H}$ has to satisfy

$$\mathsf{i}_{(X^h)^*}\Omega + \kappa(X, \delta J_\Gamma) = 0, \qquad X \in \mathfrak{X}_\mu(M)$$

for a suitable dual pair $\kappa(\mathrm{Diff}_{\mu,P}(M), \mathscr{H})$. If we choose an extension $\widehat{\omega}$ compatible with the connection, then the second term in

$$\mathsf{i}_{(X^h)^*}\Omega = (\mathsf{i}_X\mu)\,\widehat{*}\widehat{\omega} + (-1)^{\dim M}\mu\widehat{*}\left(\mathsf{i}_{(X^h)_F}\widehat{\omega}\right),$$

vanishes and we get

$$(\mathsf{i}_X\mu)\widehat{*}\widehat{\omega} + \kappa(X, \delta J_\Gamma) = 0, \qquad X \in \mathfrak{X}_\mu(M).$$

In general, it is hard to determine the 'partial' momentum map for the diffeomorphism group, mainly because one has to describe the dual group of $\mathrm{Diff}_{\mu,P}(M)$ rather concretely. For this reason, we will only study the momentum map for the most important cases of volume-preserving and symplectic diffeomorphisms.

Case 1: Momentum map for the group of volume-preserving diffeomorphisms

Recall the exact sequence

$$\mathrm{id} \longrightarrow \mathrm{Gau}(P) \longrightarrow \mathrm{Aut}_\mu(P) \longrightarrow \mathrm{Diff}_{\mu,P}(M) \longrightarrow \mathrm{id}.$$

Suppose $\mathrm{Diff}_{\mu,P}(M) = \mathrm{Diff}_\mu(M)$. We have seen that the dual group to $\mathrm{Diff}_\mu(M)$ is the group of Cheeger-Simons differential characters $\widehat{\mathrm{H}}^2(M, \mathrm{U}(1))$ parametrizing principal U(1)-bundles over M. Let $\kappa(\mathrm{Diff}_\mu(M), \widehat{\mathrm{H}}^2(M, \mathrm{U}(1)))$ be the resulting dual pair of Lie groups.

Suppose $(\underline{F}, \underline{\omega})$ has a G-equivariant prequantization U(1)-bundle $\underline{L} \to \underline{F}$ with connection $\underline{\vartheta}$. The associated U(1)-bundle

$$L := P \times_G \underline{L} \to F = P \times_G \underline{F}$$

is naturally equipped with a connection in the vertical directions. Indeed, $L := P \times_G \underline{L} \to M$ is a fiber bundle with typical fiber $\underline{L}$, so the vertical subbundle $\mathrm{V}L = P \times_G \mathrm{T}\underline{L}$ naturally projects to $\mathrm{V}F = P \times_G \mathrm{T}\underline{F}$. So the horizontal lift $\mathrm{T}\underline{F} \to \mathrm{T}\underline{L}$ induces a lift $\mathrm{V}F \to \mathrm{V}L$.

Hence a principal G-connection Γ on $P \to M$ completes $\underline{\vartheta}$ to a prequantization connection ϑ^Γ of the symplectic fiber bundle (F, ω). The connection 1-form on L is, under the identification $\mathsf{T}L = \mathsf{T}P \times_{\mathsf{T}G} \mathsf{T}\underline{L}$, given by

$$\vartheta^\Gamma_{[p,\underline{l}]}(Z^\Gamma_p + B^*_p, u_{\underline{l}}) := \underline{\vartheta}_{\underline{l}}(B^*_{\underline{l}} + u_{\underline{l}}),$$

Z^Γ_p is a Γ-horizontal vector at the point $p \in P$, $B \in \mathfrak{g}$, $u_{\underline{l}} \in \mathsf{T}_{\underline{l}}\underline{L}$. By construction, ϑ^Γ coincides with ϑ on the vertical part and hence the curvature F_{ϑ^Γ} of ϑ^Γ is an extension of the fiber symplectic structure ω. We will thus set $\widehat{\omega} = F_{\vartheta^\Gamma}$. Let $h_\Gamma \in \widehat{\mathsf{H}}^2(F, \mathsf{U}(1))$ denote the holonomy bundle of ϑ^Γ seen as a differential character.

The pull-back map $\mathsf{pb}_h : \underline{F} \ni \phi \mapsto \phi^* h_\Gamma \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$, has logarithmic derivative (formula from hat calculus)

$$(\delta\, \mathsf{pb}_h)_\phi Y = \phi^*(\mathsf{i}_Y F_{\vartheta^\Gamma} \circ \phi) = \phi^*(\mathsf{i}_Y \widehat{\omega} \circ \phi).$$

On the other hand (another formula from hat calculus), evaluate $(\mathsf{i}_X \mu)\, \widehat{*} \widehat{\omega}$ on $Y \in \mathsf{T}_\phi \underline{F}$ to obtain

$$((\mathsf{i}_X \mu)\, \widehat{*} \widehat{\omega})_\phi (Y) = (-1)^{\dim M - 1} \int_M (\mathsf{i}_X \mu) \wedge \phi^* (\mathsf{i}_Y \widehat{\omega} \circ \phi)\,.$$

Thus we have

$$(\mathsf{i}_X\mu)\,\widehat{*}\widehat{\omega} = \kappa(X, \delta\,\mathsf{pb}_h),$$

hence 'partial' momentum map is $J_\Gamma := -\mathsf{pb}_h : \underline{F} \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$.

Now we apply the fundamental theorem about momentum maps of group extensions and its corollaries.

Momentum map in the volume-preserving case. Let $\pi : P \to M$ be a finite-dimensional principal G-bundle over the closed volume manifold (M, μ) and assume that $\mathsf{Diff}_{\mu,P}(M) = \mathsf{Diff}_\mu(M)$.

1.) For every Hamiltonian G-manifold $(\underline{F}, \underline{\omega}, \underline{J})$, the section space $\mathscr{F}$ of the associated symplectic fiber bundle $(F = P \times_G \underline{F}, \omega)$ is a symplectic Fréchet manifold with weak symplectic form $\Omega = \mu\widehat{*}\omega$.

2.) Γ principal connection on $P \to M$. Suppose, that $(\underline{F}, \underline{\omega})$ has a G-equivariant prequantization U(1)-bundle $\underline{L}$ and denote the resulting prequantization of (F, ω) by $(L, \vartheta^\Gamma) = h_\Gamma \in \widehat{\mathsf{H}}^2(F, \mathsf{U}(1))$. $\mathsf{Aut}_\mu(P)$ (whose induced base diffeomorphisms are volume-preserving) acts symplectically on $(\mathscr{F}, \Omega)$ and has momentum map

$$\mathscr{J}_{\mathsf{Aut}} : \underline{F} \to \mathfrak{gau}^*(P) \times \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)), \qquad \phi \mapsto (J_* \phi, -\phi^* h_\Gamma).$$

This momentum map is equivariant relative to the natural actions.

3.) A lift $\chi : \mathsf{Diff}_\mu(M) \to \mathsf{Aut}_\mu(P)$ (a section of the projection in exact sequence centered at $\mathsf{Aut}(P)$) yields a symplectic group action of $\mathsf{Diff}_\mu(M)$ on $\mathscr{F}$ with $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$-valued momentum map

$$\mathscr{J}_{\mathsf{Diff}} : \mathscr{F} \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)), \qquad \phi \mapsto -\phi^* h_\Gamma + \iota \circ \tau^*_{\Gamma,\chi} \circ J_*(\phi),$$

$\iota : \widehat{\mathfrak{h}}^2(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ is the natural inclusion of topological trivial bundles, $\tau^*_{\Gamma,\chi} : \mathfrak{gau}^*(P) \to \widehat{\mathfrak{h}}^2(M, \mathsf{U}(1))$ is the adjoint of $\tau_{\Gamma,\chi} := \chi - \Gamma : \mathfrak{X}_\mu(M) \to \mathfrak{gau}(P)$ with respect to the dual pairs $\kappa(\mathfrak{X}_\mu(M), \widehat{\mathfrak{h}}^2(M, \mathsf{U}(1)))$ and $\langle \mathfrak{gau}(P), \mathfrak{gau}^*(P) \rangle_{\mathsf{Ad}}$. Furthermore, $\mathscr{J}_{\mathsf{Diff}}$ does not depend on the connection Γ used in the construction.

Computation of $\tau_{\Gamma,\chi} := \chi - \Gamma : \mathfrak{X}_\mu(M) \to \mathfrak{gau}(P)$ and its adjoint $\tau^*_{\Gamma,\chi}$: $\mathfrak{gau}^*(P) \to \widehat{\mathfrak{h}}^2(M, \mathsf{U}(1))$ for $\mathsf{SL}(n,\mathbb{R})$-frame bundle π: $\mathsf{L}M \to M$

$\phi \in \mathsf{Diff}_\mu(M)$ lifts to a diffeomorphism $\widehat{\phi} : \mathsf{L}M \ni u_m \mapsto \mathsf{T}_m\phi \circ u_m \in \mathsf{L}M$, where a frame $u_m \in \mathsf{L}_m M$ is an isomorphism $u_m : \mathbb{R}^n \to \mathsf{T}_m M$. Thus $\mathsf{L}M \to M$ has a natural lift $\chi : \mathsf{Diff}_\mu(M) \to \mathsf{Aut}_\mu(\mathsf{L}M)$. We get an induced lift for vector fields: $\mathfrak{X}_\mu(M) \ni X \mapsto \hat{X} \in \mathfrak{aut}_\mu(\mathsf{L}M)$.

The natural 1-form $\theta \in \Omega^1(\mathsf{L}M, \mathbb{R}^n)$ given by

$$\theta(Z_u) := \left(u^{-1} \circ \mathsf{T}_u \pi\right)(Z_u), \quad Z_u \in \mathsf{T}_u(\mathsf{L}M),$$

is invariant under naturally lifted vector fields, i.e., $\mathfrak{L}_{\widehat{X}}\theta = 0$.

$\alpha \in \Omega^1(\mathsf{L}M, \mathfrak{sl}(n,\mathbb{R}))$ principal connection, Γ the associated horizontal lift. The associated covariant derivative on $\mathsf{T}M \to M$ is

$$\nabla_Y X := u \circ \widehat{Y}^\Gamma_u(\theta(\hat{X})), \quad u \in \mathsf{L}M,$$

where $\widehat{Y}^\Gamma$ is the horizontal lift of $Y \in \mathfrak{X}(M)$.

The torsion $t_\Gamma \in \Omega^2(\mathsf{L}M, \mathbb{R}^n)$ of Γ is the covariant derivative of θ. The structure equation allows us to write

$$t_\Gamma(Z_1, Z_2) = \mathrm{d}\theta(Z_1, Z_2) + \alpha(Z_1) \cdot \theta(Z_2) - \alpha(Z_2) \cdot \theta(Z_1),$$

where $\cdot$ in the last two terms is the action of $\mathfrak{sl}(n, \mathbb{R})$ on $\mathbb{R}^n$. Evaluate this expression for $Z_1 = \hat{X} \in \mathfrak{X}(\mathsf{L}M)$, the natural lift of $X \in \mathfrak{X}(M)$, and $Z_2 = \hat{Y}^\Gamma \in \mathfrak{X}(\mathsf{L}M)$, the horizontal lift of $Y \in \mathfrak{X}(M)$:

$$\begin{aligned} t_\Gamma(\hat{X}, \hat{Y}^\Gamma) &= \mathrm{d}\theta(\hat{X}, \hat{Y}^\Gamma) + \alpha(\hat{X}) \cdot \theta(\hat{Y}^\Gamma) \\ &= -\mathrm{i}_{\hat{Y}^\Gamma}\mathrm{d}(\mathrm{i}_{\hat{X}}\theta) + \alpha(\hat{X}) \cdot u^{-1}(Y) \\ &= -u^{-1}(\nabla_Y X) + \alpha(\hat{X}) \cdot u^{-1}(Y). \end{aligned}$$

On the other hand,

$$\tau_{\Gamma,\chi}(X) = \alpha(\tau_{\Gamma,\chi}(X)) = \alpha(\hat{X} - \hat{X}^\Gamma) = \alpha(\hat{X}).$$

Combining both equations, we get

$$\begin{aligned} u(\tau_{\Gamma,\chi}(X) \cdot u^{-1}(Y)) &= u \circ t_\Gamma(\hat{X}, \hat{Y}^\Gamma) + \nabla_Y X \\ &= \nabla_X Y - [X, Y]. \end{aligned}$$

For torsion-free connections, identify $\tau_{\Gamma,\chi}$ with covariant derivative

$$\tau_{\Gamma,\chi} : \mathfrak{X}_\mu(M) \to \Gamma^\infty(\mathsf{T}^*M \otimes \mathsf{T}M), \quad X \mapsto \nabla X.$$

For a section s of $\mathsf{T}M \otimes \mathsf{T}^*M \to M$ the covariant derivative ∇s is a section of $(\mathsf{T}M \otimes \mathsf{T}^*M) \otimes \mathsf{T}^*M \to M$. We obtain a 1-form $c(\nabla s)$ by contracting the first with the last component (i.e., the vector component with the "derivative" component; in Penrose's abstract index notation $c(\nabla s)_i = \nabla_j s^j{}_i$). The identity

$$\mathrm{div}(\mathrm{i}_X s) = \langle \nabla X, s \rangle + \mathrm{i}_X c(\nabla s)$$

yields the following expression for the adjoint operator of $\tau_{\Gamma,\chi}$:

$$\tau^*_{\Gamma,\chi} : \Gamma^\infty(\mathsf{T}M \otimes \mathsf{T}^*M) \to \Omega^1(M)/\mathrm{d}\Omega^0(M), \quad s \mapsto -[c(\nabla s)].$$

Link with Donaldson's [2003] results. $\mathscr{J}_{\mathsf{Diff}} : \mathscr{F} \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ composed with the curvature map $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \to \Omega^2(M, \mathbb{R})$ yields

$$\mathscr{F} \to \Omega^2(M, \mathbb{R}), \qquad \phi \mapsto -\phi^* \widehat{\omega} + \mathsf{d}(\tau^*_{\Gamma,\chi} \circ J_*(\phi)),$$

Donaldson's "momentum map" (Eq. (8) in Donaldson [2003]). Note that $\Omega^2(M, \mathbb{R})$ is not the dual of the Lie algebra $\mathfrak{X}_\mu(M)$ but just the dual to the exact volume-preserving vector fields.

We determined a group-valued momentum map for the $\mathsf{Diff}_\mu(M)$-action under weak assumptions on $(\underline{F}, \underline{\omega})$. *A classical momentum map exists only under certain topological conditions on the fiber bundle F.* A necessary and sufficient condition for the existence of a momentum map $J : \mathscr{F} \to (\mathfrak{X}_\mu(M))^*$ is the vanishing of the obstruction map $\mathfrak{X}_\mu(M)/[\mathfrak{X}_\mu(M), \mathfrak{X}_\mu(M)] \ni [X] \longmapsto [\mathsf{i}_{X^*}\Omega] \in \mathsf{H}^1(\mathscr{F}, \mathbb{R})$.

Suppose $\exists \chi : \mathsf{Diff}_\mu(M) \to \mathsf{Aut}_\mu(P)$ lift. Then there is an induced lift for vector fields $\mathfrak{X}_\mu(M) \ni X \mapsto \hat{X} \in \mathfrak{aut}_\mu(P)$. Assume that $\hat{X}$ is horizontal with respect to a principal connection on $P \to M$ compatible with the extension $\widehat{\omega}, \forall X \in \mathfrak{X}_\mu(M)$, and that $\mathsf{Diff}_{\mu,P}(M) = \mathsf{Diff}_\mu(M)$. These assumptions are satisfied for trivial bundle $F = M \times \underline{F} \to M$.

In this setting, the the obstruction to the existence of a momentum map $J : \mathscr{F} \to (\mathfrak{X}_\mu(M))^*$ takes the form

$$\mathsf{H}^{\dim M-1}(M,\mathbb{R}) \ni [\alpha] \longmapsto [\alpha]\widehat{*}[\widehat{\omega}] \in \mathsf{H}^1(\mathscr{F},\mathbb{R}),$$

where we used the isomorphism $\mathfrak{X}_\mu(M)/[\mathfrak{X}_\mu(M),\mathfrak{X}_\mu(M)] \ni [X] \longmapsto [\mathrm{i}_X\mu] \in \mathsf{H}^{\dim M-1}(M,\mathbb{R})$. We do not know what are the minimal conditions guaranteeing that the above obstruction vanishes.

Topological data contained in the group-valued momentum map. The $\widehat{\mathsf{H}}^2(M,\mathsf{U}(1))$-valued momentum map contains two types of information. First, the curvature essentially corresponds to the classical momentum map (previous remark).

Second, the group-valued momentum map also yields topological classes, which are not detectable by the classical momentum map. There is a short exact sequence ($\Omega^1_{\mathrm{cl},\mathbb{Z}}(M)$ = closed 2-forms with integral periods, τ = inclusion, c = Chern class)

$$0 \longrightarrow \Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M) \xrightarrow{\tau} \widehat{\mathsf{H}}^2(M,\mathsf{U}(1)) \xrightarrow{c} \mathsf{H}^2(M,\mathbb{Z}) \longrightarrow 0.$$

The Chern class

$$c(\mathscr{J}_{\mathsf{Diff}}(\phi)) = -\phi^* c(h_{\mathsf{\Gamma}})$$

can even be viewed as an obstruction to the existence of a classical momentum map. Indeed, if a classical momentum map with values in $\Omega^1(M)/\mathrm{d}\Omega^0(M)$ exists, then its composition with the inclusion $\tau : \Omega^1(M)/\mathrm{d}\Omega^0(M) \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ (see exact sequence) is a group-valued momentum map whose Chern class vanishes because the image of τ lies in the kernel of the Chern class map c.

If the Chern class is trivial, then a secondary topological class related to the equivariance of the momentum map appears. This secondary class has its origin in the short exact sequence above. Exactness implies that the kernel of the characteristic class map c is isomorphic to $\Omega^1(M)/\Omega^1_{\mathsf{cl},\mathbb{Z}}(M)$. However, this isomorphism is not canonical and involves a choice. For example, we may fix piece-wise smooth loops γ_i, which determine a basis of the singular homology $\mathsf{H}_1(M, \mathbb{Z})$.

The curvature of a differential character $h \in \hat{\mathsf{H}}^2(M, \mathsf{U}(1))$ with trivial Chern class is an exact 2-form, say $\mathrm{curv}_h = \mathrm{d}\alpha_h$. The primitive α_h is uniquely determined, modulo $\Omega^1_{\mathrm{cl},\mathbb{Z}}(M)$, by the requirement that $\exp\left(2\pi\mathrm{i}\int_{\gamma_i}\alpha_h\right) = h(\gamma_i)$ for all i. By construction, the map $\tau_\gamma^{-1} : h \mapsto [\alpha_h]$ is inverse to τ and thus yields the claimed isomorphism $\ker c \to \Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M)$ of Abelian Lie groups.

Given a $\hat{\mathsf{H}}^2(M, \mathsf{U}(1))$-valued momentum map $\mathscr{J}$, whose image is contained in $\ker c$, composition $\mathscr{J}_\gamma = \tau_\gamma^{-1} \circ \mathscr{J}$ is a $\Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M)$-valued momentum map, i.e., almost a classical momentum map.

Note that, even if $\mathscr{J}$ is equivariant with respect to pull-backs by diffeomorphisms, the reduced momentum map $\mathscr{J}_\gamma$ is no longer equivariant since the loops γ_i are also affected by the diffeomorphisms. The obstruction for the equivariance is the class in $\mathsf{H}^1(M, \mathsf{U}(1))$ given by evaluating h on the generators γ_i.

We note that the same construction goes through in the slightly more general case where the Chern class does not vanish but is torsion does (because then the curvature is an exact form).

We return to this class later, where we recover, as an example, the Liouville class of a Lagrangian embedding.

Summary: If a $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$-valued momentum map exists, then the existence of a classical equivariant momentum map is obstructed by two topological classes: the Chern class in $\mathsf{H}^2(M, \mathbb{Z})$ and a class in $\mathsf{H}^1(M, \mathsf{U}(1))$.

Case 2: Momentum map for the group of symplectomorphisms

Mcompact connected boundaryless $2n$-dimensional manifold endowed with a symplectic form σ. The Liouville volume form is $\mu_\sigma = \sigma^n/n!$. Consider the exact sequence

$$\mathsf{id} \to \mathsf{Gau}(P) \to \mathsf{Aut}_\sigma(P) \to \mathsf{Diff}_{\sigma,P}(M) \to \mathsf{id},$$

where $\mathsf{Aut}_\sigma(P)$ is the group of automorphisms of the bundle $P \to M$ whose induced diffeomorphisms on the base preserve σ. Assume $\mathsf{Diff}_{\sigma,P}(M) = \mathsf{Diff}_\sigma(M)$. The momentum map is determined similarly to the volume-preserving case. Indeed, we only need to replace the dual pair $\kappa(\mathsf{Diff}_\mu(M), \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)))$ by the dual pair $\kappa(\mathsf{Diff}_\sigma(M), \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)))$ found earlier.

Suppose σ prequantizable. Choose a prequantization circle bundle with connection $h_\sigma \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$. Assume that $(\underline{F}, \underline{\omega})$ has a G-equivariant prequantization bundle $\underline{L}$. A principal connection Γ in $P \to M$ yields a prequantization $h_\Gamma \in \widehat{\mathsf{H}}^2(F, \mathsf{U}(1))$ of the symplectic fiber bundle (F, ω).

We claim that the 'partial' momentum map $J_\Gamma : \mathscr{F} \to \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1))$ assigns to a section ϕ the degree $2n$-bundle $-\phi^* h_\Gamma \star h_\sigma^{n-1}$, where the star product extends the wedge product to differential characters and the power $n-1$ should also be understood in this sense. In order to verify that the 'partial' momentum map relation

$$(\mathsf{i}_X \mu_\sigma) \widehat{*} \widehat{\omega} + \kappa(X, \delta J_\Gamma) = 0, \qquad X \in \mathfrak{X}_\sigma(M)$$

holds, we first observe that J_Γ is the composition of the pull-back map pb_{h_Γ} with the Lie group homomorphism

$$\widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)), \quad h \mapsto h \star h_\sigma^{n-1}.$$

The corresponding Lie algebra homomorphism is given by $\alpha \mapsto \alpha \wedge \sigma^{n-1}$ since by the properties of the $\star$-product of differential characters

$$\tau(\alpha) \star h_\sigma^{n-1} = \tau(\alpha \wedge \mathsf{curv} h_\sigma^{n-1}).$$

By the distribution law of the logarithmic derivative over Lie group homomorphisms (Proposition II.4.1.1 in Neeb [2006]), we get

$$\begin{aligned}\kappa(X, \delta_\phi J_\Gamma(Y)) &= -\frac{(-1)^{\dim M - 1}}{(n-1)!}\int_M (\mathsf{i}_X\sigma)\wedge\phi^*(\mathsf{i}_Y\widehat{\omega}\circ\phi)\wedge\sigma^{n-1}\\ &= -(-1)^{\dim M-1}\int_M(\mathsf{i}_X\sigma)\wedge\frac{\sigma^{n-1}}{(n-1)!}\wedge\phi^*(\mathsf{i}_Y\widehat{\omega}\circ\phi)\\ &= -(-1)^{\dim M-1}\int_M(\mathsf{i}_X\mu_\sigma)\wedge\phi^*(\mathsf{i}_Y\widehat{\omega}\circ\phi),\\ &= -((\mathsf{i}_X\mu_\sigma)\widehat{*}\widehat{\omega})_\phi(Y),\end{aligned}$$

where the last equality follows from the definition of the hat-product.

Momentum map in the symplectic case. $\pi : P \to M$ finite-dimensional principal G-bundle over the compact connected boundaryless $2n$-dimensional symplectic manifold (M, σ). Assume that $\mathrm{Diff}_{\sigma,P}(M) = \mathrm{Diff}_\sigma(M)$ in the exact sequence above. Let $\mu_\sigma := \sigma^n/n!$ be the Liouville volume form on M.

1.) Suppose σ has a prequantization $h_\sigma \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$. For every Hamiltonian G-manifold $(\underline{F}, \underline{\omega}, \underline{J})$, the space of sections $\mathscr{F}$ of the associated symplectic fiber bundle $(F = P \times_G \underline{F}, \omega)$ is a symplectic Fréchet manifold with symplectic form $\Omega = \mu_\sigma \widehat{*} \omega$.

2.) Γ principal connection on $P \to M$. Suppose that $(\underline{F}, \underline{\omega})$ has a G-equivariant prequantization $\mathsf{U}(1)$-bundle $\underline{L}$. Let $(L, \vartheta^\Gamma) =: h_\Gamma \in \widehat{\mathsf{H}}^2(F, \mathsf{U}(1))$ be the resulting prequantization of (F, ω). $\mathsf{Aut}_\sigma(P)$ (induced base diffeomorphisms preserve σ) acts symplectically on $(\mathscr{F}, \Omega)$ and has momentum map

$$\mathscr{J}_{\mathsf{Aut}} : \mathscr{F} \to \mathfrak{gau}^*(P) \times \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)), \qquad \phi \mapsto (J_* \phi, -\phi^* h_\Gamma \star h_\sigma^{n-1}).$$

This momentum map is equivariant relative to the natural actions.

3.) A lift $\chi : \mathsf{Diff}_\sigma(M) \to \mathsf{Aut}_\sigma(P)$ of the group of symplectomorphisms to principal bundle automorphisms (this is a section of the projection in the exact sequence above) yields a symplectic group action of $\mathsf{Diff}_\sigma(M)$ on $\mathscr{F}$ with $\widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1))$-valued momentum map

$$\mathscr{J}_{\mathsf{Diff}} : \mathscr{F} \to \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)), \qquad \phi \mapsto -\phi^* h_\Gamma \star h_\sigma^{n-1} + \iota \circ \tau^*_{\Gamma,\chi} \circ J_*(\phi).$$

Here $\iota : \widehat{\mathfrak{h}}^{2n}(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1))$ is the natural inclusion of topological trivial bundles. $\tau^*_{\Gamma,\chi} : \mathfrak{gau}^*(P) \to \widehat{\mathfrak{h}}^{2n}(M, \mathsf{U}(1))$ is the adjoint of $\tau_{\Gamma,\chi} : \mathfrak{X}_\sigma(M) \to \mathfrak{gau}(P)$ with respect to the dual pairs $\kappa(\mathfrak{X}_\sigma(M), \widehat{\mathfrak{h}}^{2n}(M, \mathsf{U}(1)))$ and $\langle \mathfrak{gau}(P), \mathfrak{gau}^*(P) \rangle_{\mathsf{Ad}}$. $\mathscr{J}_{\mathsf{Diff}}$ does not depend on the connection Γ that was used in the construction.

Link with Donaldson's [1999] results. Let $F = M \times \underline{F} \to M$ be a trivial bundle. Then the term involving $\tau_{\Gamma,\chi}$ vanishes. After composing the momentum map $\mathscr{J}_{\mathsf{Diff}} : \mathscr{F} \to \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1))$ with the curvature $\widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)) \to \Omega^{2n}(M, \mathbb{R})$ we get the map

$$\mathscr{F} \to \Omega^{2n}(M, \mathbb{R}), \quad \phi \mapsto -\phi^* \widehat{\omega} \wedge \sigma^{n-1}.$$

Let H_ϕ be the smooth map on M determined by

$$H_\phi\, \mu_\sigma = \phi^* \widehat{\omega} \wedge \sigma^{n-1}.$$

Donaldson [1999], Proposition 3, found that the map $\phi \mapsto H_\phi$ is the momentum map for the action of the symplectomorphism group under certain topological conditions on M. However, we remark again that $\Omega^{2n}(M, \mathbb{R})$ or $C^\infty(M)$ is not the dual of the whole Lie algebra of symplectic vector fields $\mathfrak{X}_\sigma(M)$ but just the dual to the space of Hamiltonian vector fields.

Note that the existence of a momentum map in the classical sense is again obstructed by a map on the Abelianization of $\mathfrak{X}_\sigma(M)$. The map $X \mapsto \mathrm{i}_X \sigma$ identifies $\mathfrak{X}_\sigma(M)/[\mathfrak{X}_\sigma(M), \mathfrak{X}_\sigma(M)]$ with the cohomology group $\mathrm{H}^1(M, \mathbb{R})$.

APPLICATIONS

Trivial bundle case

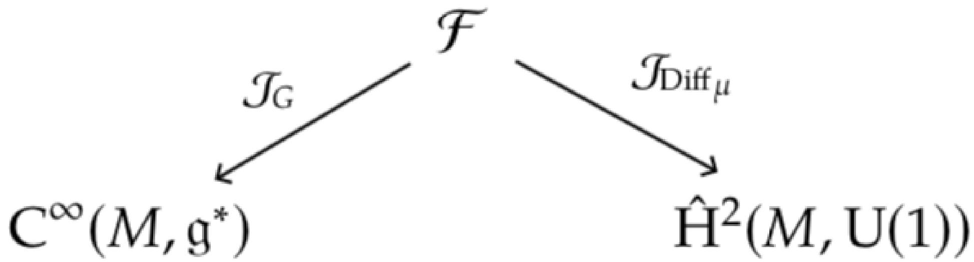
Let M be a compact connected boundaryless manifold, μ a volume form on M. We are interested in the space of smooth maps $\mathscr{F} = C^\infty(M,F)$ from M to a given Hamiltonian G-manifold (F,ω,J). This set-up fits in the general theory by thinking of $\mathscr{F}$ as the space of sections of the trivial bundle $M\times F\to M$. The symplectic form on $\mathscr{F}$ is expressed in this case as

$$\Omega_\phi(Y_1,Y_2) = \int_M \omega_\phi(Y_1,Y_2)\mu.$$

We consider the two natural actions

$$C^\infty(M,G)\times\mathscr{F}\to\mathscr{F}, \qquad \psi\cdot\phi = \psi\cdot_G\phi,$$
$$\mathrm{Diff}_\mu(M)\times\mathscr{F}\to\mathscr{F}, \qquad \varphi\cdot\phi = \phi\circ\varphi^{-1},$$

where $\cdot_G$ denotes group multiplication in G. Suppose that (F,ω) has an G-equivariant prequantum bundle (L,ϑ). Then the above actions have momentum maps and the results of the theorem about the momentum map for the group of volume preserving diffeomorphisms is summarized in the following diagram



where the momentum maps are computed to be

$$\mathscr{J}_G : \mathscr{F} \to C^\infty(M, \mathfrak{g}^*), \qquad \phi \mapsto J \circ \phi,$$
$$\mathscr{J}_{\mathsf{Diff}_\mu} : \mathscr{F} \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)), \qquad \phi \mapsto -\phi^*[L, \vartheta].$$

If ω is exact, say $\omega = \mathsf{d}\vartheta$, then $\mathscr{J}_{\mathsf{Diff}_\mu}$ is a standard momentum map

$$\mathscr{J}_{\mathsf{Diff}_\mu} : \mathscr{F} \to \Omega^1(M)/\mathsf{d}\Omega^0(M), \qquad \phi \mapsto -[\phi^*\vartheta].$$

Example 1: Hydrodynamics

Conclusion: The Lie group-valued momentum map can be used to construct a generalized Clebsch representation of a vector field that cannot be expressed in terms of classical Clebsch variables.

Geometric setting. M compact 3-dimensional orientable boundaryless Riemannian manifold, μ Riemannian volume form. Euler's equation for an incompressible perfect fluid on M is

$$\partial_t v + \nabla_v v = -\operatorname{grad} p$$

for $v \in \mathfrak{X}(M)$ subject to the incompressibility condition $\mathfrak{L}_v \mu = 0$ (i.e, $\operatorname{div}_\mu v = 0$). By taking the Riemannian dual and using the identity

$$\mathfrak{L}_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2}\mathrm{d}\|v\|^2$$

we can express Euler's equation as

$$\partial_t \alpha + \mathfrak{L}_v \alpha = \mathrm{d}\tilde{p}, \quad v^\flat = \alpha, \quad \tilde{p} := -p + \frac{1}{2}\|v\|^2.$$

The Hodge decomposition $\Omega^1(M) = \ker d^* \oplus \operatorname{im} d$ (which is L^2-orthogonal) shows that the projection $\mathrm{pr}_{\ker \mathrm{d}^*}$ onto the kernel of the codifferential induces an isomorphism between divergence free vector fields and $\Omega^1(M)/\mathrm{d}\Omega^0(M)$. The resulting isomorphism

$$A : \mathfrak{X}_\mu(M) \ni v \mapsto \left[v^\flat\right] \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$$

is called the *inertia operator*.

So Euler's equation is a dynamical system on $\Omega^1(M)/\mathrm{d}\Omega^0(M)$:

$$\partial_t[\alpha] + \mathfrak{L}_v[\alpha] = 0, \quad A(v) = [\alpha].$$

These equations are a Lie-Poisson system (Marsden-Weinstein [1983]) for the Hamiltonian

$$\mathcal{H}([\alpha]) = \frac{1}{2}\int_M \|A^{-1}[\alpha]\|^2 \mu.$$

Clebsch variables give a parametrization of $[\alpha] \in \Omega^1(M)/\mathrm{d}\Omega^0(M)$ in such a way that Euler's equation are cast into a symplectic Hamiltonian form. Geometric construction (Marsden and Weinstein [1983], Gay-Balmaz and Vizman [2012]). Let $(F, \omega = \mathrm{d}\vartheta)$ symplectic, $\mathscr{F} := C^\infty(M, F)$. As we have seen previously,

$$\mathscr{J} : \mathscr{F} \to \Omega^1(M)/\mathrm{d}\Omega^0(M), \qquad \phi \mapsto -[\phi^*\vartheta]$$

is equivariant momentum map for the $\mathrm{Diff}_\mu(M)$-action on $\mathscr{F}$. Since equivariant momentum maps are Poisson, $\mathscr{J}$ maps solutions of the Hamiltonian system $(\mathscr{F}, \Omega, \mathcal{H} \circ \mathscr{J})$ to solutions of Euler's equation. For v, $\mathscr{J}$ gives representation in terms of Clebsch variables:

$$v^\flat = -\mathrm{pr}_{\ker \mathrm{d}^*}\phi^*\vartheta = -\phi^*\vartheta + \mathrm{d}f,$$

($f \in C^\infty(M)$ ensures that $\mathrm{div}_\mu f = 0$; such an f always exists by the Hodge decomposition $\Omega^1(M) = \ker d^* \oplus \mathrm{im}\, d$).

For example, in the case $F = \mathbb{R}^2$ with $\vartheta = -x\mathrm{d}y$ we obtain

$$v^\flat = \phi_1 \mathrm{d}\phi_2 + \mathrm{d}f,$$

which is the classical Clebsch representation for the velocity field (Clebsch [1857, 1859]).

Helicity. If v admits a representation in terms of Clebsch variables, then its helicity has to vanish. Indeed, since $\dim F = 2$ and $\partial M = \varnothing$,

$$\mathsf{Hel}(v) := \int_M v^\flat \wedge \mathsf{d}v^\flat = \int_M \phi^*(\vartheta \wedge \mathsf{d}\vartheta) = 0.$$

Helicity measures the linkage and/or knottiness of vortex lines. Hence the helicity constraint implies that many topological non-trivial vector fields do not admit a Clebsch representation.

Generalized Clebsch representation for vector fields with integral helicity via group-valued momentum maps. Suppose (F, ω) not exact but admits a prequantum bundle $L \to F$ with connection ϑ. We saw that the group-valued momentum map is then

$$\mathscr{J} : \mathscr{F} \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \qquad \phi \mapsto -\phi^*[L, \vartheta].$$

Suppose $\exists\phi_0 \in \mathscr{F}$ such that $\phi_0^* L$ is trivial. Then $\forall\phi \in \mathscr{F}_{\phi_0}$, connected component of ϕ_0, bundle $\phi^* L$ is trivial since the composition

$$\mathscr{F} \xrightarrow{\mathscr{I}} \hat{\mathsf{H}}^2(M, \mathsf{U}(1)) \xrightarrow{c} \mathsf{H}^2(M, \mathbb{Z})$$

is a continuous map. Hence restriction $\mathscr{I}\restriction_{\mathscr{F}_{\phi_0}}$ takes values in

$$c^{-1}(0) = \Omega^1(M)/\Omega^1_{\mathsf{cl},\mathbb{Z}}(M).$$

By duality, we get the following *generalized Clebsch representation* of a vector field

$$v^\flat = -\phi^* \vartheta + \nu,$$

where $\nu \in \Omega^1_{\mathsf{cl},\mathbb{Z}}(M)$ and, with a slight abuse of notation, $\phi^*\vartheta$ denotes the 1-form on M that is induced by the pull-back connection $\phi^*\vartheta$ in the trivial bundle $\phi^* L$.

Euler's equation descend to $\Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M)$. In fact, the Lie derivative of a one-form β with integral periods is exact since

$$\int_\gamma \mathfrak{L}_X\beta = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \underbrace{\int_{F^X_\varepsilon(\gamma)}\beta}_{\in\mathbb{Z}} = 0, \quad F^X_\varepsilon \text{ flow of } X,$$

for every closed curve γ. Thus Euler's equation on $\Omega^1(M)/\mathrm{d}\Omega^0(M)$ implies that the class $[\beta] \in \Omega^1_{\mathrm{cl},\mathbb{Z}}(M)/\mathrm{d}\Omega^0(M) \simeq \mathrm{H}^1(M,\mathbb{Z})$ is constant in time. Thus the interesting part of the motion takes place in $\Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M)$. However, it is only possible to define an inertia operator $\Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M) \to \mathfrak{X}_\mu(M)$ for small times since the principal bundle $\mathrm{H}^1(M,\mathbb{R}) \to \mathrm{H}^1(M,\mathbb{R})/\mathrm{H}^1(M,\mathbb{Z})$ is only locally trivial.

To show that the helicity of a vector field admitting a generalized Clebsch representation has to be an integer, we start with the following general remarks about the helicity. The helicity is

$$\mathrm{Hel} : \Omega^1(M) \ni \alpha \mapsto \int_M \alpha \wedge \mathrm{d}\alpha \in \mathbb{R}.$$

Note that Hel vanishes on closed one-forms. If one forgets the hydrodynamical context for a moment, then one would mistake such an expression as the Abelian Chern-Simons functional of a trivial bundle. The similarity between helicity and Chern-Simons theory was already pointed out in Liu and Ricca [2012]. The Chern-Simons functional admits a well-known generalization to non-trivial circle bundles (e.g., Eq. 1.28, Freed, Moore, Segal [2006]). For a differential character $h \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$, the square $h \star h \in \widehat{\mathsf{H}}^4(M, \mathsf{U}(1))$ is a group homomorphism $Z_3(M, \mathbb{Z}) \to \mathsf{U}(1)$. We can evaluate it, in particular, on the 3-cycle M. Writing the evaluation as integration in order to keep the similarity with theformula for the helicity above, we obtain the *generalized helicity* (or *generalized Chern-Simons functional*)

$$\mathsf{Hel} : \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \ni h \mapsto \int_M h \star h \in \mathsf{U}(1).$$

The properties of the star product yield the following commutative diagram connecting the ordinary helicity with its generalized version

$$\begin{array}{ccc} \Omega^1(M)/\Omega^1_{\mathrm{cl},\mathbb{Z}}(M) & \xrightarrow{\mathrm{Hel}} & \mathbb{R} \\ \downarrow \tau & & \downarrow \exp \\ \hat{\mathrm{H}}^2(M,\mathrm{U}(1)) & \xrightarrow{\mathrm{Hel}} & \mathrm{U}(1). \end{array}$$

We finally observe that the generalized helicity of elements in the image of the group-valued momentum map vanish, due to a dimension count similar to the one leading to the identity that showed the vanishing of the standard helicity, namely,

$$\begin{aligned}(\mathrm{Hel}\circ \mathscr{J})(\phi) &= \mathrm{Hel}(-\phi^*[L,\vartheta]) = \int_M (\phi^*[L,\vartheta]) \star (\phi^*[L,\vartheta]) \\ &= \int_M \phi^*([L,\vartheta]\star[L,\vartheta]) = 1.\end{aligned}$$

This reasoning, of course, does not imply that the ordinary helicity of $\phi^*[L,\vartheta]$ has to vanish, only that it should be an integer.

Example: For concreteness, let $M = S^3$ and consider $F = S^2$ with its usual normalized volume ω. We take the Hopf bundle (L, ϑ) as the prequantization of (S^2, ω). Since $\mathrm{H}^2(S^3, \mathbb{R}) = 0$, the pull-back of L by every map $\phi : S^3 \to S^2$ is trivial. Many interesting and topological non-trivial knots on S^3 can be constructed from the pull-back of ϑ by a special choice of ϕ. For example, the Hopf vector field (Example III.1.9 in Arnold and Khesin [1998]) corresponds to the classical Hopf fibration

$$\phi : \mathbb{C}^2 \supseteq S^3 \ni (z_1, z_2) \mapsto [z_1 : z_2] \in \mathbb{CP}^1 \simeq S^2.$$

Other knots that can be realized in this way include the figure of eight knot, linked rings, and the trefoil knot, among others (see Kedia Foster EtAl [2016] for details). The helicity of such vector fields coincides, of course, with the Hopf invariant of ϕ. These knots have, in general, an integral but non-vanishing helicity. Hence they *do not* admit a classical Clebsch representation, but have, by construction, a generalized Clebsch representation.

The following example suggests that not every vector field with integral helicity admits a generalized Clebsch representation and that one might need to replace the symplectic manifold F by a singular symplectic stratified space.

Example: Consider the vector field $v = A \sin z\, \partial_x + A \cos z\, \partial_y$ in 3 dimensions. Since v is 2π-periodic in all directions of space, it naturally lives on the torus $M = \mathbb{T}^3$; v is a special case of an Arnold–Beltrami–Childress (ABC) flow with $B = 0 = C$ in the usual notation. In particular, v provides an example for a simple steady-state solution of Euler's equations. The vorticity is given by $\omega = A \cos z \mathrm{d}z \wedge \mathrm{d}x + A \sin z \mathrm{d}y \wedge \mathrm{d}z$, so that the helicity is

$$\mathrm{Hel} = \int_{\mathbb{T}^3} v^\flat \wedge \omega = \int_{\mathbb{T}^3} A^2 \mathrm{vol} = A^2 \mathrm{vol}(\mathbb{T}^3).$$

Seen as a vector field on $\mathbb{R}^3$, we can represent v as

$$v^\flat = f \mathrm{d}g + \mathrm{d}h, \qquad \begin{cases} f = y \sin z - x \cos z, \\ g = z, \\ h = x \sin z + y \cos z. \end{cases}$$

The pair (f, g) of functions is $\mathbb{Z}^3$-equivariant with respect to the natural action on $\mathbb{R}^3$ and the twisted action on $\mathbb{R}^2$: the group element $(n_1, n_2, n_3) \in \mathbb{Z}^3$ acts by

$$\mathbb{R}^2 \ni (u, v) \mapsto (2\pi n_2 \sin v - 2\pi n_1 \cos v, v + 2\pi n_3) \in \mathbb{R}^2.$$

Note that the action is not free and the orbit space $F = \mathbb{R}^2/\mathbb{Z}^3$ has singularities.

Example 2: Isotropic and Lagrangian embeddings

Conclusion: The moduli space of Lagrangian embeddings can be realized as a symplectic reduction by slightly modifying the previous set-up.

(S, μ) compact connected boundaryless manifold with volume form μ, (M, ω) symplectic manifold. Since the space of smooth embeddings of S into M is an open subset of $C^\infty(S, M)$, the symplectic form Ω restricts to $\mathsf{Emb}(S, M)$. As above, we assume that (M, ω) has a prequantization bundle (L, ϑ). Since the momentum map for the action of $\mathsf{Diff}_\mu(S)$ is given by

$$\mathscr{J}_{\mathsf{Diff}_\mu} : \mathsf{Imm}(S, M) \to \hat{\mathsf{H}}^2(S, \mathsf{U}(1)), \qquad \iota \mapsto -\iota^*[L, \vartheta],$$

the inverse image of the normal subgroup of all flat circle bundles on S consists precisely of isotropic embeddings.

For $\dim(S) = \dim M/2$, the set $\mathscr{J}^{-1}_{\mathsf{Diff}_\mu}(\text{flat})$ corresponds to Lagrangian embeddings. The curvature $\iota^*\omega$ of $\iota^*[L, \vartheta]$ vanishes for an isotropic embedding, by assumption. Nonetheless, the Chern class of $\iota^*[L, \vartheta]$ still may be non-trivial, although it has to be completely torsion. We are not aware that this torsion class in $\mathsf{H}^2(S, \mathbb{Z})$ is discussed anywhere in the literature.

If $\omega = \mathrm{d}\vartheta$, then the group-valued momentum map $\mathscr{J}_{\mathsf{Diff}_\mu}$ contains secondary topological data that correspond to the well-known Liouville class. Indeed, in this case, the line bundle (L, ϑ) is trivial so that the momentum map takes values in the subgroup of topological trivial bundles. Hence ι^*L is completely characterized by its curvature $\iota^*\omega$ and a class in the torus $\mathsf{H}^1(S, \mathbb{R})/\mathsf{H}^1(S, \mathbb{Z})$. For a Lagrangian embedding this class equals $[\iota^*\vartheta]$, that is, the *Liouville class* of the embedding ι (page 30, Bates and Weinstein [1997]).

an embedding is called *prequantizable* if its Liouville class vanishes (Definition 4.4, Bates and Weinstein [1997]). More generally, we say that an embedding is *torsion-free* if the torsion class in $\mathrm{H}^2(S, \mathbb{Z})$ discussed above vanishes. Summarizing, we arrive at the following correspondence:

Moduli space of	Symplectic reduction $\mathscr{J}^{-1}_{\mathrm{Diff}_\mu}(\cdot)/\,\mathrm{Diff}_\mu(S)$ at
Lagrangian embeddings	flat bundles
Torsion-free Lagrangian embeddings	flat and trivial bundles
Prequantizable Lagrangian embeddings	0

Here, we are a bit sloppy with the notion "moduli space". Since we only take the quotient by the group $\mathrm{Diff}_\mu(S) \subseteq \mathrm{Diff}(S)$, the resulting reduced space still remembers the volume form. Hence, being more precise, we obtain the space of isotropic (or Lagrangian) weighted submanifolds of type (S, μ) as the symplectic quotient.

Gay-Balmaz and Vizman [2016] showed that the quotient of isotropic embeddings modulo volume-preserving diffeomorphisms is indeed a symplectic manifold and its connected components correspond to coadjoint orbits of Ham(M, ω), at least in the case where the first cohomology group of S vanishes. It would be of interest to study the symplectic quotient without this topological assumption and search for relations with coadjoint orbits of the whole symplectomorphism group. In light of our refined results, we conclude:

Conjecture: The moduli space of weighted prequantizable Lagrangian embeddings is a symplectic manifold whose connected components are related to the coadjoint orbits of the symplectomorphism group.

Moreover, it is an interesting question whether the moduli space of (torsion-free) Lagrangian embeddings is also symplectic. A systematic study along the same lines as Gay-Balmaz and Vizman [2016] requires to develop the dual pair picture for group-valued momentum maps.

Reduction of structure group

Geometric objects usually correspond to the reduction of a principal G-bundle to a Lie subgroup $H \subseteq G$. In this setting, the fiber model $\underline{F} = G/H$. A section ϕ of the associated bundle $F = P \times_G (G/H)$ results in a reduction of $\pi : P \to M$ to the principal H-bundle

$$Q_\phi := \{p \in P \mid \phi(\pi(p)) = [p, eH]\} \to M.$$

Preparation: Pull-back of prequantum bundles as associated bundles

Conclusion: $\phi^* L \simeq Q_\phi \times_\rho \mathrm{U}(1)$ as principal U(1)-bundles over M with connections.

Suppose that $(\underline{F} = G/H, \underline{\omega}, \underline{J})$ is a Hamiltonian G-space. Evaluating $\underline{J}$ at the point $eH \in G/H$ yields a Lie algebra homomorphism

$$\underline{J}(eH)\restriction_{\mathfrak{h}} : \mathfrak{h} \to \mathbb{R}.$$

Kostant [1970]: G-equivariant prequantizations over $G/H \stackrel{\sim}{\longleftrightarrow}$ characters $\rho : H \to \mathsf{U}(1)$ integrating $\mathscr{J}(eH)\restriction_{\mathfrak{h}}$.

Indeed, given a prequantization bundle $\underline{L}$, such a character can be extracted from the circle action in the fiber over eH, that is,

$$\underline{l} \cdot h = \rho(h) \cdot \underline{l}, \quad \underline{l} \in \underline{L}_{eH}.$$

Conversely, a Lie group morphism $\rho : H \to \mathsf{U}(1)$ yields a G-equivariant prequantization bundle with connection: $\underline{L} = G \times_\rho \mathsf{U}(1) \to G/H$, G-action on $\underline{L}$ is $g \cdot [a, z] = [ga, z]$, and the connection 1-form is

$$\vartheta_{[a,z]}([a \cdot A, v]) = \langle \mathscr{J}(eH), A\rangle + v, \quad A \in \mathfrak{g}, \quad v \in \mathsf{T}_z\mathsf{U}(1) \simeq \mathbb{R}.$$

Now we prove the isomorphism $\phi^* L \simeq Q_\phi \times_\rho \mathsf{U}(1)$. Since the left G- and the right H-actions on G commute, we can identify the bundle $L = P \times_G \underline{L} \to F$ as the quotient of $P \times G \times \mathsf{U}(1)$ by the simultaneous actions of G and H, namely,

$$(p, a, z) \cdot (g, h) := (p \cdot g, g^{-1}ah, \rho(h^{-1})z).$$

From this viewpoint, the projection $L \to F$ maps $[p, a, z]$ to $[p, aH]$.

So, for a section ϕ of $F \to M$, the pull-back bundle is identified with

$$\phi^* L = \{(m, [p, a, z]) \in M \times L \mid \phi(m) = [p, aH]\}.$$

The circle principal bundle map

$$\phi^* L \to Q_\phi \times_\rho \mathsf{U}(1), \qquad (m, [p, a, z]) \mapsto [p \cdot a, z].$$

has an inverse

$$Q_\phi \times_\rho \mathsf{U}(1) \to \phi^* L, \qquad [q, z] \mapsto (\pi(q), [q, e, z])$$

so these maps are U(1)-principal bundle isomorphisms over M, where Q_ϕ is the reduced H-bundle.

Let Γ be a principal connection on P and assume that it reduces to a connection on Q_ϕ. Then the pull-back of the connection ϑ^Γ from L to $\phi^* L$ coincides with the induced connection $\rho_* \Gamma$ on the associated bundle $Q_\phi \times_\rho \mathsf{U}(1)$ under the above identification $\phi^* L \simeq Q_\phi \times_\rho \mathsf{U}(1)$.

Kähler geometry

(M, σ) symplectic $2n$-dimensional manifold. Study $\mathcal{I}$, almost complex structures compatible with σ. Each compatible almost complex structure on M gives rise to an almost Kähler manifold structure on M and thus reduces the symplectic frame bundle $\mathsf{L}M \to M$ to an $\mathsf{U}(n)$-bundle. Hence $\mathcal{I}$ is identified with the space of sections of $\mathsf{L}M \times_{\mathsf{Sp}(2n,\mathbb{R})} \underline{F}$ with fiber model $\underline{F} = \mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n)$. This homogeneous space $\underline{F}$ can be identified with the Siegel upper half space and thus is a symplectic manifold.

Get additional insight: Ohsawa [2015] *Siegel upper half space is a symplectic quotient.* $\mathsf{M}(2n \times 2n, \mathbb{R}) = \{$ real $2n \times 2n$ matrices $\}$ endowed with the constant symplectic form

$$\omega(X, Y) = \mathrm{tr}(X^{\mathsf{T}} \mathbb{J} Y), \qquad \mathbb{J} := \begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{bmatrix}$$

• Left multiplication on $\mathsf{M}(2n \times 2n, \mathbb{R})$ by $\mathsf{Sp}(2n, \mathbb{R})$
• Right multiplication on $\mathsf{M}(2n \times 2n, \mathbb{R})$ by the inverse of $\mathsf{O}(2n)$
leave the symplectic form invariant and admit momentum maps

$$\begin{gathered} \mathfrak{sp}(2n, \mathbb{R}) \xleftarrow{J_{\mathsf{Sp}}} \mathsf{M}(2n \times 2n, \mathbb{R}) \xrightarrow{J_{\mathsf{O}}} \mathfrak{o}(2n) \\ -XX^{\mathsf{T}}\mathbb{J} \longleftarrow\!\shortmid X \longmapsto X^{\mathsf{T}}\mathbb{J}X, \end{gathered}$$

duals are identified with the Lie algebras using the pairing $\frac{1}{2}\mathrm{tr}(XY^{\mathsf{T}})$. $\mathsf{U}(n)$ is the stabilizer of $\mathbb{J}$ under the (co)adjoint action of either $\mathsf{Sp}(2n, \mathbb{R})$ or $\mathsf{O}(2n)$. So identify the symplectic quotients at $\mathbb{J}$ with

$$\begin{aligned} J_{\mathsf{O}}^{-1}(\mathbb{J})/\mathsf{O}(2n)_{\mathbb{J}} &= \mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n), \\ J_{\mathsf{Sp}}^{-1}(\mathbb{J})/\mathsf{Sp}(2n, \mathbb{R})_{\mathbb{J}} &= \mathsf{O}(2n)/\mathsf{U}(n). \end{aligned}$$

This duality on the level of momentum maps and symplectic quotients correspond to the two ways to get a Kähler manifold: start from a symplectic manifold or Riemannian manifold and then choose a compatible complex structure.

Using the momentum maps, determine the prequantizations.

Recall that equivariant prequantizations of G/H are in bijective correspondence with group characters $\rho : H \to \mathsf{U}(1)$ integrating the Lie algebra homomorphism $J(eH)\restriction_{\mathfrak{h}} : \mathfrak{h} \to \mathbb{R}$.

In our case, $J_{\mathsf{O}}(e\mathsf{U}(n)) = -J_{\mathsf{Sp}}(e\mathsf{U}(n)) = \mathbb{J}$, so restriction to $\mathfrak{u}(n)$ yields the Lie algebra homomorphism $\mp\frac{1}{2}\mathsf{Tr}(\mathbb{J}\,\cdot) : \mathfrak{u}(n) \to \mathbb{R}$, which equals $\pm\mathsf{Tr}_{\mathbb{C}} : \mathfrak{u}(n) \to \mathrm{i}\mathbb{R}$ when $\mathfrak{u}(n)$ is viewed as a subalgebra of complex matrices. Thus the integrating group character is given by

$$\rho_{\pm} : \mathsf{U}(n) \to \mathsf{U}(1), \qquad A \mapsto (\det_{\mathbb{C}} A)^{\pm 1}.$$

In our sign convention, $\rho_{\pm}$ integrates $\pm\mathbb{J}$. Thus ρ_{+} corresponds to the action of $\mathsf{O}(2n)$, while ρ_{-} integrates the action of $\mathsf{Sp}(2n, \mathbb{R})$.

We have all ingredients to determine the momentum map for the action of the group of symplectomorphisms.

• (M, σ) compact symplectic $2n$-manifold.

• Identify $\mathcal{I} = \{$almost complex structures compatible with $\sigma\}$ with the space of sections of $F = \mathsf{L}M \times_{\mathsf{Sp}(2n,\mathbb{R})} \underline{F}$, whose fiber model $\underline{F} = \mathsf{Sp}(2n, \mathbb{R})/\mathsf{U}(n)$ is a symplectic manifold.

• General theory $\Rightarrow \mathcal{I}$ symplectic manifold, group of symplectomorphisms acts symplectically on it.

• $\forall I \in \mathcal{I}$, let Γ_I be the Levi-Civita connection, viewed as a connection on the reduced $\mathsf{U}(n)$-frame bundle $\mathsf{L}_I M$. Denote the extended connection on the symplectic frame bundle also by Γ_I.

• Recall that the momentum map given in the general theorem involves the pull-back of the prequantum bundle $(L, \vartheta^{\Gamma_I}) \to F$ by I (the latter viewed as a section of F). Since, by definition, Γ_I is compatible with the reduction I, we may apply the general proposition about induced connections on prequantum bundles to obtain an identification of the prequantum bundle with connection

$$(\phi^* L, \phi^* \vartheta^{\Gamma_I}) \simeq (\mathsf{L}_I M \times_{\rho_-} \mathsf{U}(1), (\rho_-)_* \Gamma_I).$$

• Group character $\rho_- : \mathsf{U}(n) \to \mathsf{U}(1)$ is the (inverse of) the determinant $\Rightarrow \mathsf{L}_I M \times_{\rho_-} \mathsf{U}(1) = \mathsf{K}_I^{-1} M$ with its Chern connection determined by the almost complex structure I.

Sidenote: I almost complex structure on M. Complex line bundle $\Lambda^{n,0}M$ of holomorphic forms is called the *canonical line bundle* and its dual $\Lambda^{0,n}M$ the *anti-canonical line bundle*. The volume form $\frac{\sigma^n}{n!}$ is a non-vanishing section of $\Lambda^{n,n}M = \Lambda^{n,0}M \otimes \Lambda^{0,n}M$ and thus can be viewed as a Hermitian metric on the (anti-)canonical bundle. The associated Hermitian frame bundles $\mathsf{K}_I M$ and $\mathsf{K}_I^{-1}M$ are principal U(1)-bundles, called the *canonical*, *anti-canonical circle bundles*. $\Diamond$

• Adjoint of $\tau_{\Gamma_{I,\chi}} : \mathfrak{X}_\sigma(M) \to \mathfrak{gau}(\mathsf{L}M)$ appears in the construction of the momentum map. However, $\tau^*_{\Gamma_{I,\chi}}$ involves the covariant derivative and thus vanishes since, by assumption, I is parallel with respect to Γ_I. Get an important special case of the general theorem.

If (M, σ) has a prequantum bundle $h_\sigma \in \hat{\mathsf{H}}^2(M, \mathsf{U}(1))$, then the symplectic action of $\mathsf{Diff}_\sigma(M)$ on the space of compatible almost complex structures $\mathcal{I}$ has an $\hat{\mathsf{H}}^{2n}(M, \mathsf{U}(1))$-valued momentum map

$$\mathscr{J}_{\mathsf{Diff}} : \mathcal{I} \to \hat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)), \qquad I \mapsto -\mathsf{K}_I^{-1}M \star h_\sigma^{n-1},$$

where $\mathsf{K}_I^{-1}M$ is the anti-canonical bundle, viewed as an element of $\hat{\mathsf{H}}^2(M, \mathsf{U}(1))$.

Comments:

1.) By definition, Ric_I is the curvature of the anti-canonical bundle $\mathsf{K}_I^{-1}M$. When I is integrable, then Ric_I is the Ricci form of the associated Kähler metric. For the curvature, we find thus

$$\mathsf{curv} \circ \mathscr{J}_{\mathsf{Diff}} : \mathscr{I} \to \Omega^{2n}(M, \mathsf{U}(1)), \qquad I \mapsto -\mathsf{Ric}_I \wedge \sigma^{n-1} = -\frac{S_I}{n}\sigma^n,$$

where $S_I = \mathsf{Tr}_\sigma \mathsf{Ric}_I$ is the (Chern-)scalar curvature. This expression was already found by Donaldson [1997], Proposition 9.

As in previous examples, the curvature momentum map is not the momentum map for the full symplectomorphism group but only for the group of exact symplectomorphisms.

The group-valued momentum map contains additional topological information: the Chern class of the bundle $\mathsf{K}_I^{-1}M \star h_\sigma^{n-1}$ is given by the cup-product $c_I(M) \cup c(h_\sigma)^{n-1}$, where $c_I(M) := c(\mathsf{K}_I^{-1}M) \in \mathsf{H}^2(M, \mathbb{Z})$ is the first Chern class of the complex manifold (M, I). In real cohomology, this class simplifies to $c_I(M) \cup [\sigma]^{n-1}$.

This formula shows that the curvature of the momentum map is essentially the scalar curvature and hence the symplectic quotients yield moduli spaces of almost Kähler metrics with prescribed scalar curvature.

Details: if $c \in \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1))$, $\mathscr{J}_{\mathsf{Diff}}^{-1}(c) = \{$all almost complex structures $I \mid \mathsf{K}_I^{-1}M \star h_\sigma^{n-1} = -c\}$. Big exact sequence for differential characters $\Rightarrow c + \widehat{\mathsf{H}}^{2n-1}(M, \mathsf{U}(1)) = \{$all circle bundles with the same curvature $\operatorname{curv} c\}$. Thus, $\mathscr{J}_{\mathsf{Diff}}^{-1}(c) = \{$all almost complex structures I with prescribed scalar curvature $S_I = -n\frac{\operatorname{curv} c}{\sigma^n}\}$.

Similarly, $c + \mathsf{Jac}^{n-1}(M)$ parametrizes circle bundles with the same curvature and topological class as c. Thus $\mathscr{J}_{\mathsf{Diff}}^{-1}(c + \mathsf{Jac}^{n-1}(M)) = \{$all almost complex structures $I \mid S_I = -n\frac{\operatorname{curv} c}{\sigma^n}$, $c_I(M) \cup c(h_\sigma)^{n-1} =$ Chern class of $c\}$.

2.) If curv$c = \lambda\sigma^n$, $\lambda \in \mathbb{R} \Rightarrow \mathscr{J}^{-1}_{\mathsf{Diff}}(c + \mathsf{Jac}^{n-1}(M)) = \{$Kähler-Einstein metrics$\}$. Indeed, an (almost) complex structure I whose Ricci form satisfies the Einstein relation $\mathsf{Ric}_I = \lambda\sigma$ has constant scalar curvature $S_I = n\lambda$. Conversely, every *integrable* complex structure I with constant scalar curvature and whose Chern class satisfies $c_I(M) = \lambda[\sigma]$ in real cohomology is automatically Kähler–Einstein because in this case, Ric_I and $\lambda\sigma$ are two harmonic forms representing the same cohomology class and thus are equal.

The problem of the existence of Kähler-Einstein metrics was recently settled with the resolution of the Yau–Tian–Donaldson conjecture (see review of Donaldson [2015]). Our infinite-dimensional symplectic picture suggests that Kähler metrics with constant scalar curvature are a more natural object to study and their moduli spaces are even symplectic quotients. However, the existence problem is harder than for Kähler-Einstein metrics.

3.) It is a recurrent theme in Kähler geometry that the existence of certain special metrics (Kähler-Einstein, positive Ricci curvature, . . .) has an algebraic stability obstruction. Donaldson [1997, 2001] related this geometric-algebraic equivalence to the interplay of symplectic momentum map geometry and the geometric invariant theory on an infinite-dimensional space. However, this approach relied on the existence of a classical momentum map and thus required certain topological classes to vanish. In light of these ideas, it would be interesting to see how the group-valued momentum map interacts with geometric invariant theory. In this direction, Diez-Janssens-Neeb-Vizman show how the hat product of differential characters can be used to construct a prequantization bundle over these infinite-dimensional function spaces. However, it is not yet clear what the existence of a group-valued momentum map means in terms of prequantum geometry.

4.) $\mathsf{K}_I^{-1}M$ often plays the role of the algebraic counterpart to a geometric property. For example, a compact complex manifold admits a Kähler metric of positive Ricci curvature only if it has an ample anti-canonical bundle (by Yau's proof of the Calabi conjecture combined with Kodaira's theorem). Our general group-valued momentum map picture explains, at least in principle, the importance of the anti-canonical bundle in many important questions concerning the geometry of (almost) Kähler manifolds: $\mathsf{K}_I^{-1}M$ is the "conserved quantity" of the group of symplectomorphisms.

5.) Recall: the Siegel upper half space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ is symplectic quotient. This additional structure transfers to the space of sections and puts the moduli space of constant scalar curvature metrics in the framework of reduction by stages.

Return to the following general set-up. $(\underline{F}, \underline{\omega})$ finite-dimensional symplectic manifold. Two Lie groups G and K act symplectically on the left on $\underline{F}$. The two actions are free, proper, commute, and have momentum maps

$$\mathfrak{g}^* \xleftarrow{\underline{J}_G} \underline{F} \xrightarrow{\underline{J}_K} \mathfrak{k}^*$$

Assume that $\underline{J}_K : \underline{F} \to \mathfrak{k}^*$ is G-invariant and $\underline{J}_G$ is K-invariant.

In the Kähler setting: $\underline{F} = \mathrm{GL}(2n, \mathbb{R})$, $G = \mathrm{O}(2n)$, $K = \mathrm{Sp}(2n, \mathbb{R})$. Hence we are in the setting of commuting reduction by stages.

Let $\mu \in \mathfrak{g}^*$. General theory: symplectic reduction by G yields the first stage reduced Hamiltonian K-space $(\underline{F}_\mu, \underline{\omega}_\mu, \underline{J}_\mu)$. The momentum map of the reduced space is induced by $\underline{J}_K : \underline{F} \to \mathfrak{k}^*$ in the obvious way. Reducing once more by the remaining K-action gives a symplectic quotient that is symplectically diffeomorphic to the corresponding symplectic quotient of the product group $G \times K$.

Do the same thing in the bundle picture for the spaces of sections. Let $\pi : P \to M$ be a principal K-bundle. The spaces occurring in the symplectic reduction diagram

$$\begin{array}{ccc} \underline{J}_G^{-1}(\mu) & \hookrightarrow & \underline{F} \\ \big\downarrow /G & & \\ \underline{F}_\mu & & \end{array}$$

all carry a natural K-action such that the inclusion and projection maps are equivariant. Thus we obtain a similar diagram for the section spaces of the associated bundles

$$\begin{array}{ccc} \Gamma^\infty(P \times_K \underline{J}_G^{-1}(\mu)) & \hookrightarrow & \mathcal{F} \\ \big\downarrow /C^\infty(M,G) & & \\ \Gamma^\infty(P \times_K \underline{F}_\mu). & & \end{array}$$

$\Gamma^\infty(P \times_K \underline{F}_\mu)$ is symplectic by general theory (it is the section space of a symplectic fiber bundle). So it is natural to suspect that $\Gamma^\infty(P \times_K \underline{F}_\mu)$ is the symplectic reduction starting from $\mathscr{F}$. Show it.

Consider the action of $C^\infty(M,G)$ on $\mathscr{F} = \Gamma^\infty(P \times_K \underline{F})$ induced by the G-action on $\underline{F}$. This action is well-defined since the G and K-actions on $\underline{F}$ commute, by assumption. Although this setting is not completely the same as the action of a gauge group on the section space of an associated bundle — $C^\infty(M,G)$ is the gauge group of the trivial bundle not of P —, the general strategy carries through and yields the momentum map

$$\mathscr{J}_G : \mathscr{F} \to C^\infty(M, \mathfrak{g}^*), \qquad \phi \mapsto (J_G)_* \phi,$$

where $J_G : P \times_K \underline{F} \ni (p, \underline{f}) \longmapsto (\pi(p), \underline{J_G}(\underline{f})) \in M \times \mathfrak{g}^*$ is a bundle map over M. From here $\Rightarrow \Gamma^\infty(P \times_K \underline{F}_\mu)$ is the symplectic reduced space of $\mathscr{F}$ by $C^\infty(M,G)$ at the constant map $\mu \in C^\infty(M, \mathfrak{g}^*)$.

Kähler setting: $\underline{F} = \mathsf{GL}(2n, \mathbb{R})$, $G = \mathsf{O}(2n)$, $K = \mathsf{Sp}(2n, \mathbb{R}) \Rightarrow G$-reduced space at $\mu = \mathbb{J}$ is $\underline{F}_\mu =$ upper Siegel half space. Thus the space of sections $\mathscr{F}$ is identified with the space $\mathcal{T}^2(M)$ of (invertible) 2-tensors and the first reduced stage $\Gamma^\infty(P \times_K \underline{F}_\mu)$ coincides with the space $\mathscr{I}$ of almost complex structures compatible with the symplectic form σ.

As discussed above, the second reduction by $\mathsf{Diff}_\sigma(M)$ at the subset $c + \widehat{\mathsf{H}}^{2n-1}(M, \mathsf{U}(1))$ yields the moduli space $\mathscr{S}\mathscr{C}$ of almost complex structures I with prescribed scalar curvature. Thus we get the following reduction by stages diagram

$$\mathcal{T}^2(M) \xrightarrow{/\!\!/_{\mathbb{J}} C^\infty(M, \mathsf{O}(2n))} \mathscr{I} \xrightarrow{/\!\!/_c \mathsf{Diff}_\sigma(M)} \mathscr{S}\mathscr{C}.$$

$$\mathcal{T}^2(M) \xrightarrow[?]{/\!\!/_{\mathbb{J},c} C^\infty(M, \mathsf{O}(2n)) \times \mathsf{Diff}_\sigma(M)} \mathscr{S}\mathscr{C}$$

We believe that this diagram commutes, i.e., the two-stage reduced space $\mathscr{S}\mathscr{C}$ can be obtained by the one-shot reduction from $\mathcal{T}^2(M)$ by the product group $C^\infty(M, \mathsf{O}(2n)) \times \mathsf{Diff}_\sigma(M)$. In finite-dimensions this is true (stages reduction book). In infinite dimensions, nothing is known. Once the general theory of reduction by stages in infinite dimensions is formulated, one can state that the curved arrow is a one-shot reduction. Future work.

Teichmüller space and weighted Lagrangian subbundles

Coadjoint orbits are the canonical examples of symplectic homogeneous spaces: G finite-dimensional Lie group, $\mu \in \mathfrak{g}^*$, coadjoint orbit $\mathcal{O}_\mu := \{\mathrm{Ad}^*_{g^{-1}}\mu \mid g \in G\} \simeq G/G_\mu$ has the orbit symplectic form

$$\omega_\nu(\mathrm{ad}^*_A \nu, \mathrm{ad}^*_B \nu) = \langle \nu, [A, B] \rangle\,, \qquad \nu \in \mathcal{O}_\mu \subset \mathfrak{g}^*, \qquad A, B \in \mathfrak{g}.$$

Let M finite-dimensional compact connected boundaryless manifold carrying a G-structure, i.e., the structure group of the frame bundle $\mathrm{L}M$ is reduced to G. A section of $\mathrm{L}M \times_G (G/G_\mu)$ corresponds to a further reduction to G_μ. General theory $\Rightarrow$ {all G_μ-reductions} has a symplectic form naturally induced by the coadjoint orbit symplectic form. The momentum map for the action of the group of diffeomorphisms of M on the space of all such G_μ-structures can be calculated from the general theory. Recall that the basic theorem involved the prequantization of the fiber model.

Prequantizations of G/G_μ bijectively correspond to Lie group characters $\rho_\mu : G_\mu \to \mathsf{U}(1)$ that integrate the Lie algebra homomorphism

$$\breve{\rho}_\mu = \underline{J}(eG_\mu)\restriction_{\mathfrak{g}_\mu} : \mathfrak{g}_\mu \to \mathbb{R},$$

$\underline{J} : G/G_\mu \to \mathfrak{g}^*$ momentum map of G-action on the coadjoint orbit, which is the inclusion after the identification $G/G_\mu \simeq \mathcal{O}_\mu$. Hence

$$\breve{\rho}_\mu : \mathfrak{g}_\mu \to \mathbb{R}, \quad A \mapsto \langle \mu, A \rangle .$$

So, find characters ρ_μ which integrate the pairing with μ.

Carry out this program for the coadjoint orbits of $\mathsf{SL}(2,\mathbb{R})$. Explicitly determine them and their stabilizer subgroups $\mathsf{SL}(2,\mathbb{R})_\mu$. Even in this simple example, the geometric structures involved turn out to be of special interest: we recover the Teichmüller moduli space as a symplectic reduction; another coadjoint orbit yields a hyperbolic cousin of the classical Teichmüller space and yet another orbit is related to Lagrangian distributions endowed with a density.

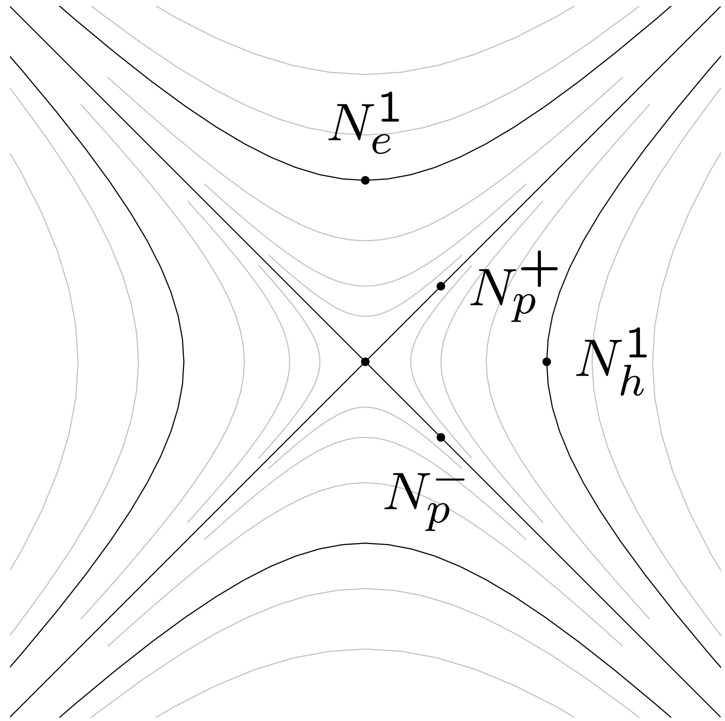
(M, σ) compact connected boundaryless surface oriented by σ (so, either symplectic or volume form).

Coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$**.** Lie algebra $\mathfrak{sl}(2,\mathbb{R}) = \{$real 2×2 matrices with vanishing trace$\}$ has the following generators

$$e_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfying $[h, e_\pm] = \mp 2e_\mp$ and $[e_+, e_-] = 2h$. The Killing form $\kappa(A,B) = \frac{1}{2}\mathrm{Tr}(AB)$ is Ad-invariant and non-degenerate, so gives an identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ and of the coadjoint and adjoint actions. $\{e_+, e_-, h\}$ is orthonormal with signature $(++-)$. Every $A \in \mathfrak{sl}(2,\mathbb{R})$, $A \neq 0$, can be brought into one of the following four normal forms under the adjoint action:

$$N_e^\lambda = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad N_h^\lambda = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad N_p^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N_p^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$



Coadjoint orbits of SL$(2,\mathbb{R})$ projected onto the e_-- h plane.

1.) Elliptic hyperboloid $\mathcal{O}_{N_e^\lambda}$ (two-sheeted hyperboloid). Elliptic orbit (eigenvalues are imaginary).

- $\lambda > 0$ parametrize upper sheets and $\lambda < 0$ lower sheets.
- $\mathsf{SL}(2,\mathbb{R})_{N_e^\lambda} = \mathsf{SO}(2)$ is independent of λ.
- $\mathcal{O}_{N_e^{\lambda=1}}$ symplectomorphic to reduced space

$$J_{\mathsf{O}}^{-1}(\mathbb{J})/\mathsf{O}(2)_{\mathbb{J}} = \mathsf{Sp}(2,\mathbb{R})/\mathsf{U}(1)$$

discussed previously (here $n = 1$).

- Sections of $\mathsf{L}M \times_{\mathsf{SL}(2,\mathbb{R})} (\mathsf{SL}(2,\mathbb{R})/\mathsf{SO}(2))$ correspond to reductions of the $\mathsf{SL}(2,\mathbb{R})$-frame bundle to $\mathsf{SO}(2)$, i.e., Riemannian metrics g compatible with the prescribed area form σ. More explicit.
- Identify $\mathcal{O}_{N_e^\lambda} \simeq \mathsf{SL}(2,\mathbb{R})/\mathsf{SO}(2)$ with the space $\mathsf{SSym}^+(2,\mathbb{R})$ of symmetric positive definite matrices having determinant 1 via

$$\mathcal{O}_{N_e^\lambda} \ni \mathsf{Ad}_g N_e^\lambda \mapsto \left(gg^{\mathsf{T}}\right)^{-1} \in \mathsf{SSym}^+(2,\mathbb{R}), \quad g \in \mathsf{SL}(2,\mathbb{R}).$$

Push forward orbit symplectic form to $\mathsf{SSym}^+(2,\mathbb{R})$:

$$\omega_B(C_1, C_2) = -\frac{1}{4}\mathsf{Tr}((B^{-1}C_1)(B^{-1}N_e^\lambda)(B^{-1}C_2)),$$

$B \in \mathsf{SSym}^+(2,\mathbb{R})$ and C_1, C_2 are traceless symmetric matrices.

2.) Hyperbolic hyperboloid $\mathcal{O}_{N_h^\lambda}$ (one-sheeted hyperboloid). Hyperbolic orbit (eigenvalues are real)

- $\mathsf{SL}(2,\mathbb{R})_{N_h^\lambda} = \mathsf{SO}(1,1)$ is independent of λ, where

$$\mathsf{SO}(1,1) = \left\{ \begin{pmatrix} \pm\cosh r & \sinh r \\ \sinh r & \pm\cosh r \end{pmatrix} \middle| \, r \in \mathbb{R} \right\}.$$

- Sections of $\mathsf{L}M \times_{\mathsf{SL}(2,\mathbb{R})} /\mathsf{SO}(1,1))$ are again reductions of the $\mathsf{SL}(2,\mathbb{R})$-frame bundle, this time, to $\mathsf{SO}(1,1)$.
- Geometrically, these are semi- (or pseudo-, in another terminology) Riemannian metrics η.

Hence the space of semi-Riemannian metrics inducing the given area form σ on M is an infinite-dimensional symplectic manifold.

3.) The light cones $\mathcal{O}_{N_p^\pm}$. Parabolic orbits (one zero eigenvalue).

- N_p^+ is upper cone, N_p^- is lower cone.
- $\mathsf{SL}(2,\mathbb{R})_{N_p^\pm} = P^\pm$, where

$$P^+ = \left\{ \begin{pmatrix} \pm 1 & r \\ 0 & \pm 1 \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}, \qquad P^- = \left\{ \begin{pmatrix} \pm 1 & 0 \\ r & \pm 1 \end{pmatrix} \,\middle|\, r \in \mathbb{R} \right\}$$

- To illuminate the geometric structure that leads to a reduction of the structure group to the stabilizer subgroups $P^\pm$, it is helpful to consider the higher dimensional case.

(M, ω) symplectic $2n$-manifold. An n-dimensional distribution D on M yields a reduction of the symplectic frame bundle to the group consisting of elements of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \quad A, C \in \mathsf{GL}(n,\mathbb{R}), \quad B \in \mathfrak{gl}(n,\mathbb{R}).$$

where $A, C \in \mathsf{GL}(n,\mathbb{R})$ and B is an arbitrary $n \times n$ matrix. Indeed, the reduced bundle consists only of frames $u_m : \mathbb{R}^{2n} \to \mathsf{T}_m M$ that map the last n standard vectors to vectors spanning D_m.

If instead require that the first n vectors span D_m, we obtain a reduction to elements of the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}.$$

If the distribution carries an additional structure, then this is reflected in further conditions on the matrix C. For example, a distribution endowed with a density $|\nu|$ leads to the condition $\det C = \pm 1$. We then say that the distribution is *weighted*.

Go back to $n = 1 \Rightarrow$ a reduction of the symplectic frame bundle to $P^{\pm}$ corresponds to a weighted Lagrangian subbundle $(L, |\nu|)$ of $\mathsf{T}M$.

Thus the space of all weighted Lagrangian distributions carries two natural symplectic structures (depending on the point $N_p^{\pm}$ chosen for the identification).

• To determine the prequantizations of the orbits, need to calculate

$$\breve{\rho}_\mu : \mathfrak{g}_\mu \to \mathbb{R}, \qquad A \mapsto \langle \mu, A \rangle$$

for each orbit. We have $\mathfrak{g}_\mu = \mathbb{R}\mu$ since $\dim \mathfrak{g}_\mu = 1$. We obtain

$$\breve{\rho}_{N_e^\lambda}(a \cdot N_e^\lambda) = -a\lambda, \qquad \breve{\rho}_{N_h^\lambda}(a \cdot N_h^\lambda) = a\lambda, \qquad \breve{\rho}_{N_p^\pm}(a \cdot N_p^\pm) = 0.$$

• **Elliptic case:** If λ satisfies the prequantization condition $\lambda \in \mathbb{Z}$, then $\breve{\rho}_{N_e^\lambda}$ integrates to the Lie group homomorphism

$$\rho_{N_e^\lambda} : \mathrm{SO}(2) \to \mathrm{U}(1), \qquad \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \mapsto e^{-\vartheta\lambda \mathrm{i}},$$

For $\lambda = 1$, this is just the inverse of the canonical identification of SO(2) with U(1). Thus, the associated bundle $\mathrm{L}_{\mathrm{SO}(2)}M \times_{\rho_{N_e^1}} \mathrm{U}(1)$ is the anti-canonical bundle $\mathrm{K}_g^{-1}M$ constructed from the Riemannian metric g. Other values of $\lambda \in \mathbb{Z}$ correspond to higher powers $\mathrm{K}_g^{-\lambda}M$.

• **Hyperbolic case:** SO$(1,1)$ is not connected and $\breve{\rho}_{N_h^\lambda}$ always admits two types of integrating characters:

$$\rho_{N_h^\lambda} : \mathsf{SO}(1,1) \to \mathsf{U}(1), \quad \begin{pmatrix} \pm\cosh r & \sinh r \\ \sinh r & \pm\cosh r \end{pmatrix} \mapsto e^{\pm r\lambda \mathrm{i}},$$

$$\sqrt{\rho_{N_h^{2\lambda}}} : \mathsf{SO}(1,1) \to \mathsf{U}(1), \quad \begin{pmatrix} \pm\cosh r & \sinh r \\ \sinh r & \pm\cosh r \end{pmatrix} \mapsto \begin{cases} e^{r\lambda \mathrm{i}} & + \\ e^{\frac{\lambda}{2}\mathrm{i}} e^{-r\lambda \mathrm{i}} & - \end{cases}$$

As the notation suggests, we have $\sqrt{\rho_{N_h^{2\lambda}}}^2 = \rho_{N_h^{2\lambda}}$. But, of course, also $\rho^2_{N_h^\lambda} = \rho_{N_h^{2\lambda}}$ holds, so that $\sqrt{\rho_{N_h^{2\lambda}}}$ is the "non-canonical square root". In analogy to the previous case, the associated bundle $\mathsf{L}_{\mathsf{SO}(1,1)}M \times_{\rho_{N_h^2}} \mathsf{U}(1)$ is the canonical bundle $\mathsf{K}_\eta M$ induced by the semi-Riemannian metric η. Then the bundles $\rho_{N_h^1}$ and $\sqrt{\rho_{N_h^2}}$ yield square roots of $\mathsf{K}_\eta M$. Thus, in contrast to the Riemannian case, also the square roots of the canonical bundle are intrinsically derived from the coadjoint orbit geometry.

• **Parabolic case:** $\check{\rho}_{N_p^\pm} = 0 \Rightarrow \rho_{N_p^\pm} : \mathsf{SL}(2,\mathbb{R})_{N_p^\pm} \ni S \mapsto 1 \in \mathsf{U}(1)$. $P^\pm$ not connected $\Rightarrow \exists$ two other non-trivial integrating characters $\rho_{N_p^\pm}$, which send $\begin{pmatrix} -1 & r \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ r & -1 \end{pmatrix}$ to $-1 \in \mathsf{U}(1)$; they factor through a homomorphism with values in $\mathbb{Z}_2$. To illuminate the significance of $\mathsf{L}_{P^\pm}M \times_{\rho_{N_p^\pm}} \mathbb{Z}_2$, it is useful to look in higher dimensions.

(M, σ) symplectic $2n$-manifold. Recall: a weighted Lagrangian distribution on M yields a reduction of the symplectic frame bundle to the structure group consisting of symplectic block matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with $C \in \mathsf{SL}^\pm(n,\mathbb{R}) := \{C \in \mathsf{SL}(n,\mathbb{R}) \mid \det C = \pm 1\}$.
The higher dimensional analog of $\rho_{N_p^+}$ is the group homomorphism

$$\rho_{N_p^+} : \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto \det C \in \mathbb{Z}_2.$$

Thus a global section of the associated flat bundle $\mathsf{L}M \times_{\rho_{N_p^+}} \mathbb{Z}_2$ corresponds to a consistent choice of orientation of the Lagrangian subspaces. So refer to the associated $\mathbb{Z}_2$-bundle as the *orientation bundle* $\mathrm{Or}L$ of L. The orientation bundle is trivial if and only if the first Stiefel–Whitney class of the Lagrangian distribution vanishes.

We summarize this discussion in the following table. It contains more; this will be explained after the theorem. We use the table, the basic theorem about the momentum maps in the volume preserving case and the proposition about pull back connections to the prequantum bundle.

μ	N_e^λ	N_h^λ	$N_p^\pm$
Type	elliptic	hyperbolic	parabolic
G_μ	SO(2)	SO(1, 1)	$P^\pm$
Quantizable	$\lambda \in \mathbb{Z}$	always	always
ρ_μ	$\rho_{N_e^\lambda}$	$\rho_{N_h^\lambda}, \sqrt{\rho_{N_h^{2\lambda}}}$	$1, \rho_{N_p^\pm}$
$\mathcal{F}_\mu$	Riemannian metric g	semi-Riemannian metric η	weighted Lagrangian distribution $(L, \lvert\nu\rvert)$
P_μ	$\mathrm{K}_g^{-1} M$	$\mathrm{K}_\eta M, \mathrm{K}_\eta^{\frac{1}{2}} M$	OrL
Characteristic class	$\mathrm{c}_1(M)$	?	$\mathrm{w}_1(L)$
Moduli space	Teichmüller space	hyperbolic Teichmüller space	?

Overview over the properties of the coadjoint orbits and the geometric structures they induce. Here, $\mathscr{F}_\mu$ denotes the space of sections of $\mathsf{L}M \times_{\mathsf{SL}(2,\mathbb{R})} (\mathsf{SL}(2,\mathbb{R})/G_\mu)$. Furthermore, P_μ is the associated circle bundle of the reduced G_μ-frame bundle using the character ρ_μ.

Let M be a compact connected boundaryless surface endowed with a symplectic form σ. Then:

1.) The space $\mathrm{Metr}_\sigma(M)$ of Riemannian metrics on M compatible with σ has the symplectic form

$$\Omega_g(h_1, h_2) = -\frac{\lambda}{4}\int_M \mathrm{Tr}(g^{-1}h_1 \cdot g^{-1}\sigma \cdot g^{-1}h_2)\,\sigma.$$

If $\lambda \in \mathbb{Z}$ then the action of $\mathrm{Diff}_\sigma(M)$ has a group-valued momentum map given by

$$\mathrm{Metr}_\sigma(M) \to \widehat{\mathrm{H}}^2(M, \mathrm{U}(1)), \quad g \mapsto \mathrm{K}_g M.$$

2.) The space $\mathsf{Metr}_\sigma^{+-}(M)$ of semi-Riemannian metrics with signature $(+-)$ and compatible with σ carries a symplectic form defined by the formula above (as for Riemannian metrics). The action of $\mathsf{Diff}_\sigma(M)$ has two group-valued momentum maps given by

$$\mathsf{Metr}_\sigma^{+-}(M) \to \hat{\mathsf{H}}^2(M, \mathsf{U}(1)), \quad \eta \mapsto \mathsf{K}_\eta^{-1} M$$
$$\mathsf{Metr}_\sigma^{+-}(M) \to \hat{\mathsf{H}}^2(M, \mathsf{U}(1)), \quad \eta \mapsto \mathsf{K}_\eta^{-\frac{1}{2}} M.$$

3.) The space of weighted Lagrangian distributions is a symplectic manifold and the group-valued $\mathsf{Diff}_\sigma(M)$-momentum map is the assignment of the orientation bundle $\mathsf{Or}L$ to each Lagrangian distribution L.

Curvature of the canonical bundle $\mathsf{K}_g M$ is $-S_I\sigma$, where S_I is the scalar curvature. Hence, symplectic reduction at the subset $\mathsf{curv}^{-1}(\sigma)$ of all bundles with constant curvature σ yields the Riemann moduli space:

$$\mathscr{J}_{\mathsf{Diff}}^{-1}(\mathsf{curv}^{-1}(\sigma))/\mathsf{Diff}_\sigma(M) = \{I \in \mathcal{I} \mid S_I = -1\}/\mathsf{Diff}_\sigma(M).$$

Instead of taking the quotient with respect to $\mathsf{Diff}_\sigma(M)$, we could restrict attention to the connected component of the identity $\mathsf{Diff}_\sigma(M)^\circ$. The action of $\mathsf{Diff}_\sigma(M)^\circ$ is free and its momentum map is given by the same formula. Thus, the reduced space coincides with $\{I \in \mathcal{I} \mid S_I = -1\}/\mathsf{Diff}_\sigma(M)^\circ$ and is hence the Teichmüller space.

The formula in 1.) for the symplectic form on the space of Riemannian metrics shows that the reduced symplectic form is proportional to the Weil-Petersson symplectic form on the Teichmüller space. However, in contrast to classical symplectic reduction, we take the inverse image of a set and not just of a point.

We show now that $\mathsf{Diff}_\sigma(M)$ acts (infinitesimally) transitively on $\mathsf{curv}^{-1}(\sigma)$, which implies that the reduction is a *symplectic orbit reduction*.

Suppose that σ is normalized to have total volume 1. Then $\exists h_\sigma \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$ with curvature σ.

Consider the flux homomorphism relative to h_σ

$$\mathsf{Flux}_{h_\sigma} : \mathsf{Diff}_\sigma(M) \to \mathsf{H}^1(M, \mathsf{U}(1)), \quad \phi \mapsto (\phi^{-1})^* h_\sigma - h_\sigma.$$

(Flux is actually only a cocycle and not necessarily a group homomorphism on $\mathsf{Diff}_\sigma(M)$. Only when restricted to $\mathsf{Diff}_\sigma(M) \cap \mathsf{Diff}(M)^\circ$ is the flux a bona-fide group homomorphism.) By smoothness of the pull-back for differential characters, the corresponding Lie algebra homomorphism

$$\mathsf{Flux}_\sigma : \mathfrak{X}_\sigma(M) \to \mathsf{H}^1(M, \mathbb{R}), \quad X \mapsto [\mathsf{i}_X \sigma],$$

is clearly surjective. Thus $\mathsf{Diff}_\sigma(M)$ acts infinitesimally transitive on $\mathsf{curv}^{-1}(\sigma)$.

The Riemann moduli space

$$\{I \in \mathcal{I} \mid S_I = -1\}/\mathsf{Diff}_\sigma(M)$$

and the Teichmüller space

$$\{I \in \mathcal{I} \mid S_I = -1\}/\mathsf{Diff}_\sigma(M)^\circ$$

are symplectic orbit reduced spaces.

Similarly, the coadjoint orbit $\mathsf{SL}(2,\mathbb{R})/\mathsf{SO}(1,1)$ yields a hyperbolic version of the Teichmüller space. Instead of Riemannian metrics, the moduli space then consists of semi-Riemannian metrics with prescribed curvature modulo symplectomorphisms.

It also carries a symplectic form analogous to the one induced by the Weil-Petersson metric and is a symplectic orbit reduced space. Formally, we just have to replace:

“elliptic“ by “hyperbolic“,

“Riemannian“ by “semi-Riemannian“

“complex structure“ by “para-complex structure“.

The analysis of hyperbolic operators is, of course, more delicate than elliptic operators and thus it is not clear that the hyperbolic Teichmüller space is indeed a smooth manifold. Physically, elements of the hyperbolic Teichmüller space parametrize equivalence classes of solutions of Einstein’s equation in two dimensions.

In the physics literature (Strobl [2000]) one finds arguments that such moduli spaces of semi-Riemannian metrics with prescribed curvature are finite dimensional, similar to the Riemannian case. We are not aware of any rigorous mathematical proof of this statement.

Action of the quantomorphism group

(M, ω) compact symplectic $2n$-manifold, ω has integral periods $\Rightarrow \omega =$ curvature of a prequantum circle bundle $\pi : P \to M$ with connection Γ. As the fiber model take $\underline{F} = \mathbb{C}$ with the usual symplectic structure $\omega_{\mathbb{C}}$ but endowed with k-times the natural $\mathsf{U}(1)$ action for some $k \in \mathbb{N}$. Thus, the associated bundle $F^k = P \times_{k \cdot \mathsf{U}(1)} \mathbb{C}$ is the k-th tensor product of the usual line bundle F^1.

The general theorem about the momentum map of the group of symplectomorphisms gives the momentum map of the action on $\Gamma^\infty(F^k)$ of the automorphism group of P that preserve the symplectic structure on the base.

Now we are only interested in the subgroup $\mathrm{Aut}_\Gamma(P)$ of automorphism that preserve the connection Γ.

Due to its importance in prequantization, $\mathrm{Aut}_\Gamma(P)$ is often called the *quantomorphism group* of (P, Γ).

Recall that its Lie algebra $\mathfrak{aut}_\Gamma(P)$ is identified with the Poisson algebra $C^\infty(M)$ using the Koszul prescription

$$C^\infty(M) \ni f \mapsto X_f^\Gamma + (\pi^* f)\partial_\vartheta \in \mathfrak{aut}_\Gamma(P),$$

- $\partial_\vartheta =$ canonical vector field along the fiber,
- $X_f^\Gamma = \Gamma$-horizontal lift of X_f.
- $\mathfrak{aut}_\omega(P) = C^\infty(M) \times \mathfrak{X}_\omega(M)$ and under this identification the inclusion $\mathfrak{aut}_\Gamma(P) \subset \mathfrak{aut}_\omega(P)$ is just

$$C^\infty(M) \ni f \longmapsto (f, X_f) \in C^\infty(M) \times \mathfrak{X}_\omega(M).$$

The dual projection (natural pairings) is

$$C^\infty(M) \times \Omega^{2n-1}(M)/\mathsf{d}\Omega^{2n-2}(M) \ni (f, [\alpha]) \longmapsto g\mu - \frac{1}{(n-1)!}\mathsf{d}\alpha \in \Omega^{2n}(M)$$

and lifts to the dual group as a map

$$C^\infty(M) \times \widehat{\mathsf{H}}^{2n}(M, \mathsf{U}(1)) \ni (f, h) \longmapsto g\mu - \frac{1}{(n-1)!}\mathsf{curv}_h \in \Omega^{2n}(M).$$

The momentum map for the quantomorphism group is the composition of this “projection“ with the momentum map for the full automorphism group given in the general theorem about the momentum map of the group of symplectomorphisms. We have hence:

$$
\begin{aligned}
\mathscr{J}_{\mathsf{Aut}_\Gamma} : \Gamma^\infty(F^k) &\to \Omega^{2n}(M) \\
\phi &\longmapsto -k\|\phi\|^2\mu + \frac{1}{(n-1)!}\phi^*\widehat{\omega}^\Gamma_{\mathbb{C}} \wedge \omega^{n-1}.
\end{aligned}
$$

Using the identity $\phi^*\widehat{\omega}^\Gamma_{\mathbb{C}} = \omega_{\mathbb{C}}(\nabla^\Gamma\phi, \nabla^\Gamma\phi) + (J_*\phi)\,\mathsf{curv}_\Gamma$ (Lemma 13 in Donaldson [2003]) we can rewrite the momentum map as follows:

$$
\begin{aligned}
&\mathscr{J}_{\mathsf{Aut}_\Gamma}(\phi) = \\
&-k\|\phi\|^2\mu + \frac{1}{(n-1)!}(\omega_{\mathbb{C}}(\nabla^\Gamma\phi, \nabla^\Gamma\phi) - k\|\phi\|^2\omega) \wedge \omega^{n-1} = \\
&-k(n+1)\|\phi\|^2\mu + \frac{\mathrm{i}}{2}\nabla^\Gamma\phi \wedge \nabla^\Gamma\bar{\phi} \wedge \frac{\omega^{n-1}}{(n-1)!}
\end{aligned}
$$

Equation 11 in Donaldson [2001] is similar (some typos).

Gauge theory

Atiyah-Bott [83]: curvature is momentum map for action of gauge group on space of connections on a surface

- Their setting fits in our general framework.
- Extend study to action of full automorphism group.
- Already know: gauge group admits an ordinary momentum map. Group-valued momentum map not needed.
- The momentum map geometry of the space of connections is also induced by a finite-dimensional symplectic system. As we shall see, the additional conserved quantity coming from the action of the full automorphism group has a purely topological character.

Atiyah-Bott [1983]: Yang-Mills equation over a Riemann surface. More generally: base is compact symplectic $2n$-manifold (M, σ); Donaldson [1987] almost same setting.

$\pi : P \to M$ principal G-bundle. Connection Γ on P is a splitting of the exact sequence of vector bundles over M

$$0 \longrightarrow \mathrm{Ad}(P) \longrightarrow (\mathsf{T}P)/G \xrightarrow{(\mathsf{T}\pi)/G} \mathsf{T}M \longrightarrow 0.$$

The bundle $\mathsf{C}P \to M$ of principal connections on P is the subbundle of $\mathsf{L}(\mathsf{T}M, (\mathsf{T}P)/G) \to M$ whose fiber $\mathsf{C}_m P = \{\Gamma_m : \mathsf{T}_m M \to (\mathsf{T}P/G)_m \text{ linear} \mid ((\mathsf{T}\pi)/G)(\Gamma_m(X_m)) = X_m, \forall X_m \in \mathsf{T}_m M\}$, $\forall m \in M$.

$\rho : \mathsf{T}P \to (\mathsf{T}P)/G$ projection. $\rho^{-1}(\Gamma_m(\mathsf{T}_m M))$ complementary subspace of the vertical subspace of $\mathsf{T}_p P$ for any $p \in \pi^{-1}(m)$; the collection of all of them forms a complementary subbundle to the vertical subbundle of $\mathsf{T}P$ which is invariant under the G-action, i.e., a horizontal G-equivariant distribution, i.e., a principal connection.

$\mathsf{Ad}P \to M$ adjoint bundle of P. If $\Lambda_m : \mathsf{T}_m M \to \mathsf{Ad}_m P$ linear $\Rightarrow \Gamma_m + \Lambda_m \in \mathsf{C}_m P$ since $\mathsf{Ad}P = \ker(\mathsf{T}\pi)/G$. So $\mathsf{C}P \to M$ is an affine bundle modeled on the vector bundle $\mathsf{L}(\mathsf{T}M, \mathsf{Ad}P) \to M$. By construction, the space of principal connections on P is identified with the space of sections of $\mathsf{C}P$.

Apply our general framework. Need $\mathsf{C}P$ = associated fiber bundle with symplectic fiber. This requires the principal jet prolongation $\mathsf{W}P$ of P. Recall it.

- The principal jet prolongation of the Lie group G is the semidirect product

$$\mathsf{W}G = (\mathsf{Sp}(2n, \mathbb{R}) \times G) \rtimes \mathsf{L}(\mathbb{R}^{2n}, \mathfrak{g})$$

with multiplication

$$(a, g, S)(b, h, T) = (ab, gh, \mathsf{Ad}_h^{-1} \circ S \circ b + T)$$

and inversion

$$(a, g, S)^{-1} = (a^{-1}, g^{-1}, -\mathsf{Ad}_g \circ S \circ a^{-1}).$$

• The principal jet prolongation $\mathsf{W}P$ is the product fiber bundle $\mathsf{L}M \times_M \mathsf{J}^1 P$ over M, where $\mathsf{L}M$ denotes the symplectic frame bundle. Equivalently,

$$\mathsf{W}_m P = \Big\{ (u_m, v_p) \mid u_m : (\mathbb{R}^{2n}, \omega_0) \to (\mathsf{T}_m M, \sigma_m) \text{ linear symplectomorphism, } v_p : \mathsf{T}_m M \to \mathsf{T}_p P \text{ linear, } \mathsf{T}_p \pi \circ v_p = \mathrm{id}_{\mathsf{T}_m M} \Big\}.$$

The right action of $\mathsf{W}G$ on $\mathsf{W}P$ defined by

$$(u_m, v_p) \cdot (a, g, S) := \left(u_m \circ a, v_p(\_) \cdot g + \left(S \circ (u_m \circ a)^{-1} \right) (\_)^* (p \cdot g) \right)$$

turns $\mathsf{W}P$ into a right principal $\mathsf{W}G$-bundle.

The second component of the action is the linear map

$$\mathsf{T}_m M \ni X_m \longmapsto v_p(X_m)\cdot g + \left(S \circ (u_m \circ a)^{-1})\right)(X_m)^*(p\cdot g) \in \mathsf{T}_{p\cdot g}P,$$

$\left(S \circ (u_m \circ a)^{-1})\right)(X_m)^*(p\cdot g)$ is the infinitesimal generator vector field of $\left(S \circ (u_m \circ a)^{-1})\right)(X_m) \in \mathfrak{g}$ evaluated at $p\cdot g$; the dot denotes the action of $g \in G$ on P and on $\mathsf{T}P$.

- $\mathsf{W}G$ acts affinely on the left on $\mathsf{L}(\mathbb{R}^{2n}, \mathfrak{g})$ by

$$(a, g, S) \cdot T = \mathsf{Ad}_g \circ (S + T) \circ a^{-1}.$$

Therefore, $\mathsf{W}P \times_{\mathsf{W}G} \mathsf{L}(\mathbb{R}^{2n}, \mathfrak{g}) = \mathsf{C}P$, bundle of connections.

The horizontal lift corresponding to is $[u_m, v_p, T]$ is

$$v_p - \left(T \circ u_m^{-1}\right)(\_)^*(p) : \mathsf{T}_m M \to \mathsf{T}_p P.$$

• Suppose $\mathfrak{g}$ has an Ad-invariant scalar product. $\mathrm{Tr}(\cdot \wedge \cdot)$ is the induced wedge product of linear maps with values in $\mathfrak{g}$. The symplectic form on $\mathsf{L}(\mathbb{R}^{2n}, \mathfrak{g})$ is given by

$$\omega(T_1, T_2) := \mathrm{Tr}(T_1 \wedge T_2) \wedge \frac{\omega_0^{n-1}}{\omega_0^n},$$

ω_0 standard symplectic form on $\mathbb{R}^{2n}$. ω is invariant under the action of $\mathsf{W}G$ since $a \in \mathrm{Sp}(2n, \mathbb{R})$ and the scalar product on $\mathfrak{g}$ is assumed to be Ad-invariant.

- The associated momentum map is given by

$$\langle J(T),(A,\xi,L)\rangle = \Big(\frac{1}{2}\mathrm{Tr}(\xi,[T\wedge T]) - \mathrm{Tr}(L\wedge T) + \frac{1}{2}\mathrm{Tr}(T\circ A\wedge T)\Big)\wedge\frac{\omega_0^{n-1}}{\omega_0^n}$$

for $A\in \mathrm{Sp}(2n,\mathbb{R})$, $\xi\in\mathfrak{g}$ and $L\in \mathrm{L}(\mathbb{R}^{2n},\mathfrak{g})$. Thus the standard fiber $\mathrm{L}(\mathbb{R}^{2n},\mathfrak{g})$ of the connection bundle is a symplectic system with Hamiltonian symmetry. In order to determine the prequantizations, note that the action of $\mathrm{W}G$ is transitive and thus $\mathrm{L}(\mathbb{R}^{2n},\mathfrak{g})$ is the homogeneous space $\mathrm{W}G/(\mathrm{Sp}(2n,\mathbb{R})\times G)$.

• Use previous strategy. Since $J(0) = 0$, WG-equivariant prequantizations over $\mathsf{L}(\mathbb{R}^{2n}, \mathfrak{g})$ are in one-to-one correspondence with Lie group homomorphisms

$$\rho : \mathsf{Sp}(2n, \mathbb{R}) \times G \to \mathsf{U}(1),$$

whose induced Lie algebra homomorphism is trivial.

• Discuss these structures on the level of bundles. A connection Γ on P is a section of C$P \to M$. Since the typical fiber of the connection bundle is a homogeneous space, we can equivalently think of a connection as a reduction of WP to the structure group $\mathsf{Sp}(2n, \mathbb{R}) \times G$.

For $p \in P$, let $\Gamma_p : \mathsf{T}_m M \to \mathsf{T}_p P$ be the horizontal lift operator. Then $P \ni p \mapsto \Gamma_p \in \mathsf{J}^1 P$ induces the reduction of structure group

$$\mathsf{L}M \times_M P \xrightarrow{\mathsf{id} \times \Gamma} \mathsf{L}M \times_M \mathsf{J}^1 P = \mathsf{W}P.$$

∇ linear connection on M compatible with the symplectic structure, which we view as a principal connection on the frame bundle $\mathsf{L}M$. The product connection $\nabla \times \Gamma$ induces a connection on the reduced bundle $R_\Gamma \subseteq \mathsf{W}P$ and thus uniquely determines a connection $\mathsf{W}_\nabla \Gamma$ on $\mathsf{W}P$ by right-invariant extension.

The general theory of momentum maps for volume preserving diffeomorphisms states that the prequantum bundle of the typical fiber induces a prequantum U(1)-bundle $L \to \mathsf{C}P$, whose connection ϑ depends on the connection $\mathsf{W}_{\nabla}\Gamma$. By the proposition about pull back connections, it follows that the pull-back of (L, ϑ) along the connection Γ (viewed as a section of $\mathsf{C}P$) coincides with the associated bundle $(\mathsf{L}M \times_M P) \times_\rho \mathsf{U}(1)$ with the connection induced by the product connection $\nabla \times \Gamma$. Denote the corresponding differential character by $h_{\nabla,\rho}(\Gamma) \in \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$. Note that $h_{\nabla,\rho}(\Gamma)$ is always flat since ρ has vanishing derivative.

The symplectic form

$$\Omega(\alpha,\beta) = \int_M \mathsf{Tr}(\alpha\wedge\beta)\wedge\sigma^{n-1} \qquad \alpha,\beta\in\Omega^1(M,\mathsf{Ad}P)$$

on the space $\mathsf{C}(P)$ of connections is invariant under the action of the group $\mathsf{Aut}_\sigma(P)$ of automorphisms that cover symplectomorphisms. If (M,σ) has a prequantization $h_\sigma \in \widehat{\mathsf{H}}^2(M,\mathsf{U}(1))$, then for every linear connection ∇ on $\mathsf{L}M$ and every Lie group homomorphism $\rho : \mathsf{Sp}(2n,\mathbb{R})\times G \to \mathsf{U}(1)$ with $\mathsf{T}_{(e,e)}\rho = 0$ the map

$$J_{\mathsf{Aut}} : \mathsf{C}(P) \to \Omega^{2n}(P,\mathsf{Ad}P)\times\widehat{\mathsf{H}}^{2n}(M,\mathsf{U}(1)),$$
$$\Gamma \mapsto \left(\mathsf{curv}_\Gamma\wedge\sigma^{n-1}, -h_{\nabla,\rho}(\Gamma)\star h_\sigma^{n-1}\right)$$

is an $\mathsf{Aut}_\sigma(P)$-group-valued momentum map.

TECHNICAL BACKGROUND

Notation and conventions

By a manifold we understand a possibly infinite-dimensional smooth manifold without boundary modeled on a locally convex space as for example discussed in Neeb [2006]. We use standard notations and conventions. If $f : M \to N$ then $\mathrm{T}f : TM \to \mathrm{T}N$ is the tangent map. $(\mathfrak{X}(M), [\cdot,\cdot])$ Lie algebra of vector fields, $\mathfrak{X}^k(M)$ are k-contravariant tensors, $[\cdot,\cdot]$ is the Schouten bracket, $\Omega^k(M)$ are the k-forms, d is the exterior differential, i_Z and $\mathfrak{L}_Z$ are the interior product with the vector field Z

and the Lie derivative in the direction Z, $\wedge$ is the wedge product of forms with the Bourbaki conventions.

WARNING: For manifolds modeled on locally convex spaces, the cotangent bundle is no longer a *smooth* bundle and hence smoothness of differential forms has to be specified explicitly. A differential form is defined as a set-theoretic section of the exterior bundle such that all chart representation are smooth as maps $U \times X^k \to \mathbb{R}$, where U is an open subset of M and X denotes the model space of M.

Given topological vector spaces V and V^* (not necessarily the functional analytic dual), $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ denotes a weakly non-degenerate pairing. If W and W^* are in weak duality and $T : V \to W$ is a linear continuous map, then its dual $T^* : W^* \to V^*$ is defined by $\langle T^* \alpha, v \rangle := \langle \alpha, Tv \rangle$, $\forall v \in V$, $\forall \alpha \in W^*$. Weak non-degeneracy of the pairings guarantees only uniqueness of T^*, not its existence, so in concrete situations, its existence needs to be checked.

For Lie groups and Lie algebras use standard notations.

If $\Upsilon : G \times M \to M$ is a left Lie group action and $A \in \mathfrak{g}$, the infinitesimal generator, or fundamental, vector field defined by A is

$$A^*_m := \left.\frac{d}{dt}\right|_{t=0} \Upsilon(\exp(tA), m) \in \mathsf{T}_m M, \quad \forall m \in M,$$

where $\exp : \mathfrak{g} \to G$ is the exponential map. We have $[A, B]^* = -[A^*, B^*]$, $\forall A, B \in \mathfrak{g}$. If the G-action on M is on the right, then $[A, B]^* = [A^*, B^*]$. We also use the dot notation: if $m \in M$, $g \in G$, $A \in \mathfrak{g}$, and $X_m \in \mathsf{T}_m M$, we abbreviate $g \cdot m := \Upsilon(g, m)$, $A^*_m = A \cdot m$, $g \cdot X_m = \mathsf{T}_m \Upsilon_g(X_m)$.

If $f : M \to G$ is a smooth function, the *left logarithmic derivative* $\delta_m f : \mathsf{T}_m M \to \mathfrak{g}$ of f at $m \in M$ is defined by $\delta_m f := \mathsf{T}_{f(m)} L_{f(m)^{-1}} \circ \mathsf{T}_m f$. Therefore $\delta f \in \Omega^1(M, \mathfrak{g})$.

Poisson manifolds, standard notations. Sign convention: $\dot{f} = \{h, f\}$ are Hamilton's equations. Canonical 2-form on T^*Q is $dp_i \wedge dq^i$.

(M, ω) symplectic G-manifold. A *momentum map* for the action, if it exists, is a smooth map $J : M \to \mathfrak{g}^*$ satisfying

$$\mathbf{i}_{A^*}\omega + \mathbf{d}J_A = 0, \quad A \in \mathfrak{g}, \quad \text{where} \quad J_A := \langle A, J \rangle : M \to \mathbb{R}.$$

$P \to M$ right principal G-bundle and $\underline{F}$ a left G-manifold. Form the *associated bundle* $F = P \times_G \underline{F} := (P \times \underline{F})/G$; free right G-action $(p, \underline{f}) \cdot g := (p \cdot g, g^{-1} \cdot \underline{f})$, for any $p \in P$, $\underline{f} \in \underline{F}$, and $g \in G$.

Special cases: $\underline{F} = \mathfrak{g}, \mathfrak{g}^*, G \Rightarrow$ Lie algebra bundle Ad$P \to M$, Lie-Poisson vector bundle Ad$^*P \to M$, conjugation group bundle Conj$P \to M$.

Let $P \to M$ be a right principal G-bundle and $\Gamma \in \Omega^1(P, \mathfrak{g})$ a principal connection. Assume that P and M are connected (hence path connected). Fix a point $p \in P$ and let $\mathrm{Hor}(\Gamma)_p \subset P$ be the set of points that can be joined to p by a piecewise smooth horizontal curve. Then $\mathrm{Hor}(\Gamma)_p \to M$ is a principal bundle, called the *holonomy bundle*, with structure group $\mathrm{Hol}(\Gamma)_p \subset G$, the holonomy group based at p, i.e., all $g \in G$ such that p and $p \cdot g$ can be joined by a piecewise smooth horizontal curve. The pull back of the connection Γ by the inclusion to $\mathrm{Hor}(\Gamma)_p$ is again a connection one-form because the Γ-parallel transport maps from fiber to fiber leave $\mathrm{Hor}(\Gamma)_p$ invariant (as a set) by construction.

Since $\text{Hol}(\Gamma)_{p \cdot g} = g^{-1}\text{Hol}(\Gamma)_p g$, $\text{Hor}(\Gamma)_{p \cdot g} = \text{Hor}(\Gamma)_p \cdot g$, and both P and M are path connected, it follows that all holonomy bundles are isomorphic over the identity on M as principal bundles. This is why we call $\text{Hor}(\Gamma)_p$ for any choice of $p \in P$ the holonomy bundle.

Fiber integration

$\pi : M \times F \to M$ trivial fiber bundle, F oriented, compact, connected, boundaryless, M could be infinite dimensional. If $\alpha \in \Omega^k(M \times F)$, the *fiber integral* is a linear map

$$\fint_F : \Omega^k(M \times F) \to \Omega^{k-\dim F}(M)$$

defined by

$$\left(\fint_F \alpha \right)_m (X_1, \ldots, X_{k-\dim F}) = \int_F \alpha_{(m,\cdot)}(X_1, \ldots, X_{k-\dim F}, \cdot).$$

Properties:

1. Equivariance with respect to maps on the base:

$$\phi^* \fint_F \alpha = \fint_F (\phi \times \mathsf{id}_F)^* \alpha, \quad \forall \phi : M \to M$$

with infinitesimal version

$$\mathfrak{L}_X \fint_F \alpha = \fint_F \mathfrak{L}_{X \times 0_F} \alpha, \quad \forall X \in \mathfrak{X}(M).$$

2. Invariance under fiber transformations: $\forall \psi : F \to F$ orientation-preserving diffeomorphisms,

$$\fint_F (\mathsf{id}_M \times \psi)^* \alpha = \fint_F \alpha,$$

with infinitesimal version $\fint_F \mathfrak{L}_{0_M \times Z} \alpha = 0$, $\forall Z \in \mathfrak{X}(F)$.

3. Contraction with vector fields:

$$\mathsf{i}_X \fint_F \alpha = \fint_F \mathsf{i}_{(X \times 0_F)} \alpha, \quad \forall Z \in \mathfrak{X}(M).$$

Furthermore, for $Z \in \mathfrak{X}(F)$, we have $\fint_F \mathsf{i}_{(0_M \times Z)} \alpha = 0$.

4. Fiber integration commutes with the exterior differential on M and $M \times F$:

$$\mathsf{d} \fint_F \alpha = \fint_F \mathsf{d}\alpha.$$

In this formula, the fact that F is boundaryless is essential. If F has a non-empty boundary, the formula above has contributions from the boundary:

$$\mathsf{d}⨍_F \alpha = ⨍_F \mathsf{d}\alpha + (-1)^{\mathsf{deg}\,\alpha + \mathsf{dim}\,F} ⨍_{\partial F} \alpha.$$

5. Up-down formula: for any $\beta \in \Omega^k(M)$ and $\alpha \in \Omega^l(M \times F)$ we have

$$⨍_F \pi^* \beta \wedge \alpha = \beta \wedge ⨍_F \alpha.$$

Hat product for fiber bundles

Vizman [2011] introduced the hat product of differential forms as a method to construct differential forms on the function space $C^\infty(M, F)$ starting from differential forms on M and F. We need to extend this method to induce differential forms on the Fréchet manifold of smooth sections $\underline{F}$ of a finite-dimensional fiber bundle $\pi : F \to M$ with M compact, connected, boundaryless, oriented.

Let ev : $M \times \underline{F} \ni (m,\phi) \mapsto \phi(m) \in F$ be the evaluation map and $\mathrm{pr}_M : M \times \underline{F} \to M$ the projection on the first factor. Define the *hat product* as the bilinear map

$$\widehat{*} : \Omega^k(M) \times \Omega^l(F) \to \Omega^{k+l-\dim M}(\underline{F})$$
$$\alpha \,\widehat{*}\, \omega = \fint_M \mathrm{pr}_M^* \alpha \wedge \mathrm{ev}^* \omega;$$

fiber integration is over the trivial bundle $M \times \underline{F} \to \underline{F}$. So the formula is only a suggestive notational convenience; recall that the exterior bundle is not a smooth bundle for Fréchet manifolds (because the cotangent bundle is not). Need to rewrite the definition as:

$$(\alpha \,\widehat{*}\, \omega)_\phi(Y_1, \dots Y_r) = (-1)^{kr} \int_M \alpha \wedge \phi^*(\mathsf{i}_{Y_r} \cdots \mathsf{i}_{Y_1}(\omega \circ \phi)),$$

where $r = k + l - \dim M$ denotes the degree of the hat product, $Y_i \in \Gamma^\infty(\phi^* \mathsf{V}F)$, and $\mathsf{V}F \subset \mathsf{T}F$ is the vertical subbundle. We used the abbreviation $\phi^*(\mathsf{i}_{Y_r} \cdots \mathsf{i}_{Y_1}(\omega \circ \phi)) \in \Omega^{l-r}(M)$ for the partial pull-back

$$\begin{aligned}
&\phi^*(\mathsf{i}_{Y_r} \cdots \mathsf{i}_{Y_1}(\omega \circ \phi))_m(X_1, \dots X_{l-r}) \\
&\qquad = \omega_{\phi(m)}(Y_1(m), \dots, Y_r(m), \mathsf{T}_m\phi(X_1), \dots, \mathsf{T}_m\phi(X_{l-r})) \\
&\qquad = (\mathrm{ev}^*\omega)_{m,\phi}(Y_1, \dots, Y_r, X_1, \dots, X_{l-r}).
\end{aligned}$$

If $\alpha = \mu$ is a volume form on M we get the following l-form on $\underline{F}$:

$$(\mu \widehat{*} \omega)_\phi(Y_1, \ldots, Y_l) = (-1)^{l \dim M} \int_M \phi^*(\mathrm{i}_{Y_l} \cdots \mathrm{i}_{Y_1}(\omega \circ \phi))\, \mu.$$

Since $Y_i(m)$ is a vertical tangent vector at $\phi(m)$, we see that $\mu \widehat{*} \omega$ only depends on the vertical component of the form ω. Hence, in this case, we can also define the hat product of μ with a differential form ω along the fibers of F by using this formula as the definition.

Hat product calculus

$\pi : F \to M$ finite-dimensional fiber bundle, $\alpha \in \Omega^k(M)$, $\omega \in \Omega^l(F)$.

1.) $\mathrm{d}(\alpha \mathbin{\widehat{*}} \omega) = (\mathrm{d}\alpha) \mathbin{\widehat{*}} \omega + (-1)^k \alpha \mathbin{\widehat{*}} (\mathrm{d}\omega)$.

In particular if $\alpha = \mu$ is a volume form on M, then

$$\mathrm{d}(\mu \mathbin{\widehat{*}} \omega) = (-1)^{\dim M} \mu \mathbin{\widehat{*}} (\mathrm{d}_\pi \omega),$$

where d_π denotes the vertical differential along the fibers.

2.) For every bundle automorphism ψ of F, denote the induced diffeomorphism on the base M by $\check{\psi}$. With this notation, the canonical smooth left action of $\mathsf{Aut}(F)$ on the section space $\underline{F}$ takes the form

$$\Upsilon : \mathsf{Aut}(F) \times \underline{F} \to \underline{F}, \qquad \psi \cdot \phi = \psi \circ \phi \circ \check{\psi}^{-1}.$$

On the infinitesimal level, we obtain for $Y \in \mathfrak{aut}(F)$,

$$Y^*_\phi = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \psi_\varepsilon \circ \phi \circ \check{\psi}_\varepsilon^{-1} = Y \circ \phi - \mathsf{T}\phi \circ \check{Y} \in \Gamma^\infty(\phi^* \mathsf{V} F),$$

where $\check{Y}$ is the induced vector field on M by Y.

3.) Suppose $\check{\psi}$ is an orientation preserving diffeomorphism of M. Then

$$\Upsilon_{\psi}^{*}(\alpha \,\widehat{*}\, \omega) = (\check{\psi}^{*}\alpha) \,\widehat{*}\, (\psi^{*}\omega),$$
$$\mathfrak{L}_{Y^{*}}(\alpha \,\widehat{*}\, \omega) = (\mathfrak{L}_{\check{Y}}\alpha) \,\widehat{*}\, \omega + \alpha \,\widehat{*}\, (\mathfrak{L}_{Y}\omega)$$
$$\mathsf{i}_{Y^{*}}(\alpha \,\widehat{*}\, \omega) = \left(\mathsf{i}_{\check{Y}}\alpha\right) \widehat{*}\, \omega + (-1)^{k}\alpha \,\widehat{*}\, (\mathsf{i}_{Y}\omega)\,.$$

Cheeger-Simons differential characters

Cheeger-Simons differential characters [1985] model connections on higher circle n-bundles. So they generalize circle bundles to “bundles” whose curvature is not a 2-form but an n-form. The idea is to adopt and generalize the representation of principal circle bundles with connection via their holonomy map. The standard reference is the book by Baer and Becker [2013]. However, we need, in addition, the derivative of the pull-back map and the generalization of the hat product to differential characters.

1.) Motivation: circle bundles

Denote the set of equivalence classes of principal U(1)-bundles with connections over M by $\widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$. We have four canonical maps:

Curvature The curvature curv_Γ of the connection Γ is a 2-form on M. The Bianchi identity says that curv_Γ is closed. Furthermore, the periods of the curvature are integral (after dividing by 2π). Thus, we get a curvature map to the space of closed 2-forms with integral periods

$$\mathsf{curv} : \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \to \Omega^2_{\mathsf{cl},\mathbb{Z}}(M, \mathbb{R}).$$

Flat bundle The holonomy of a flat connection yields a homomorphism $\mathsf{Hol} : \pi_1(M) \to \mathsf{U}(1)$. Since the target is an Abelian group, the commutator subgroup $[\pi_1(M), \pi_1(M)]$ lies in the kernel of Hol. On the other hand, by the Hurewicz theorem,

$$\pi_1(M)/[\pi_1(M), \pi_1(M)] \quad \text{is isomorphic to} \quad \mathsf{H}_1(M, \mathbb{Z}).$$

Universal Coefficient Thm. $\Rightarrow \mathsf{Hom}(\mathsf{H}_1(M, \mathbb{Z}), \mathsf{U}(1)) \simeq \mathsf{H}^1(M, \mathsf{U}(1))$ (since $\mathsf{U}(1)$ is divisible). Thus, flat line bundles are represented by $\mathsf{H}^1(M, \mathsf{U}(1))$ and we get a natural injection

$$\iota : \mathsf{H}^1(M, \mathsf{U}(1)) \simeq \mathsf{Hom}(\pi_1(M)/[\pi_1(M), \pi_1(M)], \mathsf{U}(1)) \\ \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)).$$

Characteristic class Assigning the Chern class to a circle bundle gives a map $c : \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)) \to \mathsf{H}^2(M, \mathbb{Z})$.

Trivial bundle Connections on $M \times \mathsf{U}(1) \to M$ are canonically identified with $\Omega^1(M, \mathbb{R})$. So, view $\Omega^1(M, \mathbb{R})$ as trivial bundles with connection and get a map $\tau : \Omega^1(M, \mathbb{R}) \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1))$. Since $\ker \tau = \{\alpha \in \Omega^1(M, \mathbb{R}) \mid \alpha$ gauge-equivalent to $0\}$, there exists a gauge transformation $\phi : M \to \mathsf{U}(1)$ such that $\alpha = \delta\phi$. For connected M, this is exactly the case if α is closed and integral. Thus we get an injection

$$\tau : \Omega^1(M, \mathbb{R})/\Omega^1_{\mathsf{cl},\mathbb{Z}}(M, \mathbb{R}) \to \widehat{\mathsf{H}}^2(M, \mathsf{U}(1)).$$

2.) Differential characters

Work with smooth singular (co-)homology. Let $Z_{k-1}(M,\mathbb{Z})$ be the group of smooth integral-valued singular cycles of degree $k-1$.

A group homomorphisms $h : Z_{k-1}(M,\mathbb{Z}) \to \mathrm{U}(1)$ is called a *differential character* if $\exists \mathrm{curv}_h \in \Omega^k(M,\mathbb{R})$ satisfying

$$h(\partial c) = \exp\left(2\pi i \int_c \mathrm{curv}_h\right), \ \forall \text{ smooth singular } k\text{-chain } c.$$

The k*-th Cheeger-Simons differential cohomology group* $\widehat{\mathrm{H}}^k(M,\mathrm{U}(1))$ is the subgroup of $\mathrm{Hom}(Z_{k-1}(M,\mathbb{Z}),\mathrm{U}(1))$ consisting of differential characters.

Cheeger-Simons differential characters provide a model for differential cohomology theory in the sense that we get a functor $\widehat{\mathrm{H}}^*(\cdot, \mathrm{U}(1))$ from the category of smooth manifolds to the category of $\mathbb{Z}$-graded Abelian groups, together with four natural transformations:

- curvature $\mathsf{curv} : \widehat{\mathsf{H}}^*(\cdot, \mathsf{U}(1)) \to \Omega^*_{\mathsf{cl},\mathbb{Z}}(\cdot, \mathbb{R})$

- inclusion of flat classes $\iota : \widehat{\mathsf{H}}^{*-1}(\cdot, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^*(\cdot, \mathsf{U}(1))$

- characteristic class $c : \widehat{\mathsf{H}}^*(\cdot, \mathsf{U}(1)) \to \mathsf{H}^*(\cdot, \mathbb{Z})$

- topological trivialization $\tau : \Omega^{*-1}(\cdot, \mathbb{R})/\Omega^{*-1}_{\mathsf{cl},\mathbb{Z}}(\cdot, \mathbb{R}) \to \widehat{\mathsf{H}}^*(\cdot, \mathsf{U}(1))$

These maps generalize those discussed before for circle bundles.

Moreover, for every smooth manifold M and $k \in \mathbb{Z}$ the following diagram commutes and all rows and columns are exact:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Jac}^{k-1}(M) & \longrightarrow & \dfrac{\Omega^{k-1}(M,\mathbb{R})}{\Omega^{k-1}_{\mathrm{cl},\mathbb{Z}}(M,\mathbb{R})} & \longrightarrow & \mathrm{d}\,\Omega^{k-1}(M,\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^{k-1}(M,\mathrm{U}(1)) & \xrightarrow{\iota} & \hat{\mathrm{H}}^{k}(M,\mathrm{U}(1)) & \xrightarrow{\mathrm{curv}} & \Omega^{k}_{\mathrm{cl},\mathbb{Z}}(M,\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle c} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}_{\mathrm{Ab}}(\mathrm{H}_{k-1}(M,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & \mathrm{H}^{k}(M,\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_{k}(M,\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here $\mathsf{Jac}^l(M) := \mathsf{H}^l(M,\mathbb{R})/\mathsf{H}^l(M,\mathbb{Z})$ denotes the Jacobian torus. The Abelian group $\mathsf{Ext}_{\mathsf{Ab}}(\mathsf{H}_{k-1}(M,\mathbb{Z}),\mathbb{Z})$ of equivalence classes of $\mathbb{Z}$-extensions of $\mathsf{H}_{k-1}(M,\mathbb{Z})$ is canonically identified with the torsion subgroup of $\mathsf{H}^k(M,\mathbb{Z})$.

The horizontal sequence mainly captures information about the connection. For ordinary principal U(1)-bundle it just states that every closed, integer 2-form can be realized as the curvature of a connection and that flat bundles are parametrized by their holonomy on generators of $\pi_1(M)$.

The topological information is encoded in the vertical sequences. The surjective map $c : \widehat{\mathsf{H}}^k(M, \mathsf{U}(1)) \to \mathsf{H}^k(M, \mathbb{Z})$ shows that differential cohomology refines integer-valued cohomology.

Let M be a compact manifold. For every $k \geq 1$, the group $\widehat{\mathsf{H}}^k(M, \mathsf{U}(1))$ is an infinite-dimensional Fréchet Lie group diffeomorphic to

$$\mathsf{H}^k(M,\mathbb{Z}) \times \left(\Omega^{k-1}(M,\mathbb{R})/\Omega^{k-1}_{\mathsf{cl},\mathbb{Z}}(M,\mathbb{R})\right).$$

Its Lie algebra $\widehat{\mathfrak{h}}^k(M, \mathsf{U}(1))$ is isomorphic to the Abelian Fréchet Lie algebra

$$\Omega^{k-1}(M,\mathbb{R})/\mathsf{d}\Omega^{k-2}(M,\mathbb{R}).$$

With respect to this differentiable structure, all natural maps curv, ι, c, τ are smooth and hence the defining diagram above is a commutative diagram of Lie groups.

3.) Geometric construction of the group structure of $\widehat{\mathrm{H}}^2(M, \mathrm{U}(1))$ for principal $\mathrm{U}(1)$-bundles

This is due to Kobayashi [1956]. The addition of two principal U(1)-bundles P and $\tilde{P}$ over M is defined as follows. First, consider the fiber product $P \times_M \tilde{P}$ and then identify the points which differ by an U(1)-action, i.e., we set

$$P + \tilde{P} := \left(P \times_M \tilde{P} \;/\; (p, \tilde{p}) \sim (p \cdot z, \tilde{p} \cdot z^{-1})\right)$$

$P + \tilde{P}$ is indeed a smooth principal U(1)-bundle, where the U(1)-action is the translation in the first factor. The trivial bundle constitutes the identity element, i.e., $P + (M \times \mathrm{U}(1))$ is isomorphic to P. For a given principal bundle P, we let $-P$ denote the U(1)-bundle that has the same underlying bundle structure as P but carries the modified U(1)-action

$$p * z := p \cdot z^{-1},$$

where on the right side U(1) acts as in P. Then $P + (-P)$ is isomorphic to the trivial bundle.

This addition of bundles is commutative and associative. Connections A and $\tilde{A}$ on the bundles combine to a connection $\mathrm{pr}_1^*A + \mathrm{pr}_2^*\tilde{A}$ on $P \times_M \tilde{P}$, which descends to a connection $A + \tilde{A}$ on $P + \tilde{P}$. The curvature of $A + \tilde{A}$ is the sum of the corresponding curvatures.

4.) Differential character calculus

A.) Pull-back. $f : M \to N$ smooth induces a pull-back

$$f^* : \widehat{\mathsf{H}}^k(N, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^k(M, \mathsf{U}(1)).$$

Here $f^*h := h \circ f_*$ is the dual of the induced map $f_* : Z_{k-1}(M, \mathbb{Z}) \to Z_{k-1}(N, \mathbb{Z})$ on cycles.

Let $h \in \widehat{\mathsf{H}}^k(N, \mathsf{U}(1))$. For every compact connected boundaryless manifold M, the pull-back map

$$\mathsf{pb}_h : C^\infty(M, N) \to \widehat{\mathsf{H}}^k(M, \mathsf{U}(1)), \qquad \phi \mapsto \phi^* h$$

is smooth with logarithmic derivative

$$\delta_\phi \mathsf{pb}_h(X) = \phi^*(\mathsf{i}_X \mathsf{curv}_h \circ \phi), \quad \text{for all } X \in \mathsf{T}_\phi C^\infty(M, N).$$

Need formula for Lie derivative along $\phi \in C^\infty(M, N)$ in direction of a "vector field" $X : M \to \mathsf{T}N$ over ϕ, i.e., $X \in \mathsf{T}_\phi C^\infty(M, N)$. This formula should also generalize Cartan magic formula liking $\mathfrak{L}_Y$, i_Y, and d for $Y \in \mathfrak{X}(M)$.

Let $\alpha \in \Omega^k(N)$. For every compact connected boundaryless manifold M, the pull-back map

$$\mathsf{pb}_\alpha : C^\infty(M, N) \to \Omega^k(M), \qquad \phi \mapsto \phi^*\alpha$$

is smooth and has derivative

$$\mathsf{T}_\phi \mathsf{pb}_\alpha(X) = \mathfrak{L}_X^\phi \alpha = \mathsf{d}(\phi^*(\mathsf{i}_X(\alpha \circ \phi))) + \phi^*(\mathsf{i}_X \mathsf{d}(\alpha \circ \phi)).$$

Here $\mathfrak{L}_X^{\phi}\alpha := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_\varepsilon^*\alpha$ denotes the Lie derivative along the map ϕ and $X \in \mathsf{T}_\phi C^\infty(M, N)$, so that the right hand side is the analogue of Cartan's formula. Moreover, we used the notation of the partial pull-back.

B.) Product. There exists an associative and $\mathbb{Z}$-bilinear product of differential characters denoted by

$$\star : \widehat{\mathsf{H}}^k(M, \mathsf{U}(1)) \times \widehat{\mathsf{H}}^l(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{k+l}(M, \mathsf{U}(1)).$$

It satisfies the following properties for $h \in \widehat{\mathsf{H}}^k(M, \mathsf{U}(1))$, $g \in \widehat{\mathsf{H}}^l(M, \mathsf{U}(1))$:

1. (Ring structure) The product $\star$ is associative and bilinear.

2. (Graded commutativity) $h \star g = (-1)^{kl} g \star h$.

3. (Naturality under pull-back) For a smooth map $f : N \to M$, we have $f^*(h \star g) = (f^*h) \star (f^*g)$.

4. (Compatibility with curvature) $\mathrm{curv}_{h \star g} = \mathrm{curv}_h \wedge \mathrm{curv}_g$.

5. (Compatibility with characteristic class) $c(h \star g) = c(h) \cup c(g)$.

6. (Compatibility with trivializations) $\tau(\alpha) \star h = \tau(\alpha \wedge \mathrm{curv}_h)$, where $\tau : \Omega^{*-1}(\cdot, \mathbb{R})/\Omega^{*-1}_{\mathrm{cl},\mathbb{Z}}(\cdot, \mathbb{R}) \to \widehat{\mathrm{H}}^*(\cdot, \mathrm{U}(1))$ is topological trivialization.

Baer and Becker [2013] show: these properties completely characterize the $\star$-product; lengthy proof of this statement and the explicit construction of $\star$. We need only these properties, not concrete construction of $\star$.

C.) Fiber integration. $\pi : F \to M$ smooth fiber bundle whose typical fiber $\underline{F}$ is a compact, connected, boundaryless, oriented manifold. Baer and Becker [2013] show that there is the notion of fiber integration of differential characters which yields a group homomorphism $⨍ : \widehat{\mathsf{H}}^k(F, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{k-\dim \underline{F}}(M, \mathsf{U}(1))$ such that the curvature map intertwines fiber integration of differential characters with ordinary fiber integration of differential forms $⨍ : \Omega^k(F) \to \Omega^{k-\dim \underline{F}}(M)$, i.e., the following diagram commutes

$$\begin{array}{ccc} \hat{\mathrm{H}}^k(F, \mathrm{U}(1)) & \xrightarrow{\int} & \hat{\mathrm{H}}^{k-\dim \underline{F}}(M, \mathrm{U}(1)) \\ \Big\downarrow \mathrm{curv} & & \Big\downarrow \mathrm{curv} \\ \Omega^k(F) & \xrightarrow{\int} & \Omega^{k-\dim \underline{F}}(M). \end{array}$$

5.) Hat product of differential characters

Generalize hat product to differential characters. The hat product of differential characters yields a method that combines higher bundles on M and F to bundles on the function space. Thus, we generalize also the procedure of higher dimensional transgression (see Baer and Becker [2013], Chapter 9.2).

M compact, connected, boundaryless, oriented, $\pi : F \to M$ finite-dimensional fiber bundle. Recall that the hat product of differential forms used the evaluation and projection map to construct differential forms on $M \times \underline{F}$ and then integrate out the M-factor. Same strategy to define the hat product of differential characters.

The evaluation map $\mathsf{ev} : M \times \underline{F} \to F$ yields a pull-back map of differential characters $\mathsf{ev}^* : \widehat{\mathsf{H}}^l(F, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^l(M \times \underline{F}, \mathsf{U}(1))$. Similarly, the projection on the first factor induces a homomorphism $\mathsf{pr}_M^* : \widehat{\mathsf{H}}^k(M, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^k(M \times \underline{F})$. Taking the product of these differential forms yields an element of $\widehat{\mathsf{H}}^{k+l}(M \times \underline{F})$. Finally, we integrate over M by viewing $M \times \underline{F}$ as a trivial bundle over $\underline{F}$. In summary, we have constructed a map

$$\widehat{*} : \widehat{\mathsf{H}}^k(M, \mathsf{U}(1)) \times \widehat{\mathsf{H}}^l(F, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{k+l-\dim M}(\underline{F}, \mathsf{U}(1)),$$

$$h \,\widehat{*}\, g = \fint_M \mathsf{pr}_M^* h \star \mathsf{ev}^* g.$$

Using compatibility with the curvature map, we calculate

$$\mathsf{curv}_{h \widehat{*} g} = \fint_M \mathsf{pr}_M^* \mathsf{curv}_h \wedge \mathsf{ev}^* \mathsf{curv}_g = \mathsf{curv}_h \,\widehat{*}\, \mathsf{curv}_g,$$

that is, the curvature of the hat product of differential characters equals the hat product of the corresponding curvatures.

Interesting special cases.

Prequantization Fix a differential character μ on M whose degree coincides with the dimension of M. Then we define a μ-dependent map as

$$\widehat{\mathsf{H}}^l(F, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^l(\underline{F}, \mathsf{U}(1)), \qquad h \mapsto \mu \mathbin{\widehat{*}} h.$$

In particular, for $l = 2$, principal U(1)-bundles on F yield principal U(1)-bundles on the section space $\underline{F}$. By the formula above, the bundle $\mu \mathbin{\widehat{*}} h$ is a prequantization of $(\underline{F}, \mathsf{curv}_\mu \mathbin{\widehat{*}} \mathsf{curv}_g)$.

Transgression The *transgression map* (Baer and Becker [2013], Definition 9.1) uses the hat product with the identity element $e \in \widehat{\mathsf{H}}^0(M, \mathsf{U}(1))$:

$$\widehat{\mathsf{H}}^l(F, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{l-\dim M}(\underline{F}, \mathsf{U}(1)), \quad h \mapsto e \widehat{*} h = \fint_M \mathsf{ev}^* h.$$

For the special case of a trivial fiber bundle $F = \underline{F} \times M$ over $M = S^1$, the space of smooth sections is identified with the space $\mathcal{L}(\underline{F})$ of smooth loops in $\underline{F}$ and transgression yields a map $\widehat{\mathsf{H}}^l(\underline{F}, \mathsf{U}(1)) \to \widehat{\mathsf{H}}^{l-1}(\mathcal{L}(\underline{F}), \mathsf{U}(1))$. In degree $l = 2$, it assigns to a principal $\mathsf{U}(1)$-bundle over $\underline{F}$ its holonomy map $\mathcal{L}(\underline{F}) \to \mathsf{U}(1)$.