Geometric theory of flexible and expandable tubes conveying fluid

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Review: Variational principles in mechanics

Consider the system of n variables $\boldsymbol{q} = (q_1, \ldots, q_n)$ and Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$. Equations of motion are obtained by Hamilton's critical action principle

$$\delta S = \delta \int_{t_0}^{t_1} L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mathrm{d}t = 0$$

on variations $\delta \boldsymbol{q}(t_0) = \delta \boldsymbol{q}(t_1) = 0$. Assume $\boldsymbol{q}(t) = \boldsymbol{q}_0(t) + \epsilon \delta \boldsymbol{q}(t)$ and select the first-order terms in ϵ to get Euler-Lagrange equations

$$\delta S = \delta \int_{t_0}^{t_1} L(\boldsymbol{q}_0 + \epsilon \delta \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_0 + \epsilon \delta \dot{\boldsymbol{q}}) \mathrm{d}t$$

$$= \epsilon \int \frac{\partial L}{\partial \boldsymbol{q}} \delta \boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \delta \dot{\boldsymbol{q}} \mathrm{d}t + O(\epsilon^2) = \epsilon \int \underbrace{\left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right)}_{=\mathbf{0}:\text{EL eqs}} \delta \boldsymbol{q} \mathrm{d}t + O(\epsilon^2)$$

Euler-Lagrange equations: $$-\frac{d}{dt} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} + \frac{\partial L}{\partial \boldsymbol{q}} = \mathbf{0}$$

Euler-Lagrange equations are systematic, but not always easy to use . . .

Motion of a rigid body

- Configuration manifold: orientation matrices, $\Lambda \in Q = SO(3)$, with 3×3 matrices $\Lambda^T \Lambda = \Lambda \Lambda^T = \mathrm{Id}$
- Lagrangian is $L = L(\Lambda, \dot{\Lambda})$, Euler-Lagrange equations are given by

$$\delta \int L(\Lambda, \dot{\Lambda}) \mathrm{d}t = 0 \quad \text{on matrices satisfying} \quad \Lambda^T \Lambda - \mathrm{Id} = 0$$

- Can, in principle, write Euler-Lagrange equations with Lagrange multipliers for 3×3 matrices (9 equations, 6 multipliers): awkward, see *e.g.* Jose & Saletan's book.

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- Can, in principle, write Euler-Lagrange equations with Lagrange multipliers for 3×3 matrices (9 equations, 6 multipliers): awkward, see *e.g.* Jose & Saletan's book.
- Instead, notice that $\Lambda^T \dot{\Lambda}$ is an antisymmetric matrix, define body angular velocity $\mathbf{\Omega}$ according to $(\Lambda^T \dot{\Lambda})_{ij} = \epsilon_{ijk} \Omega_k$ and write Euler's equations as angular momentum conservation law

$$\mathbb{I} \dot{\mathbf{\Omega}} = \mathbb{I} \mathbf{\Omega} \times \mathbf{\Omega} \, , \quad L = \frac{1}{2} \mathbb{I} \mathbf{\Omega} \cdot \mathbf{\Omega} \, , \quad \mathbf{\Omega} = (\Lambda^T \dot{\Lambda})^\vee \, .$$

- Here, we have explicitly assumed that the Lagrangian (kinetic energy) depends on the body angular velocity $\mathbf{\Omega} = (\Lambda^T \dot{\Lambda})^\vee$, and not on the spatial angular velocity $\boldsymbol{\omega} = (\dot{\Lambda} \Lambda^T)^\vee \neq \mathbf{\Omega}$, because the moment of inertia $\mathbb{I}$ is constant in the body frame.

Variational derivation of Euler's rigid body equations

- Configuration manifold $\Lambda \in SO(3)$, a Lie group, with its Lie algebra (tangent at unity element) of antisymmetric matrices $\widehat{\Omega} \in \mathfrak{so}(3)$.
- System is invariant wrt (left) rotations: $L(R\Lambda, R\dot{\Lambda}) = L(\Lambda, \dot{\Lambda})$
- Identify 3×3 antisymmetric matrices from $\mathfrak{so}(3)$ and vectors with cross products, $\widehat{\Omega} = \mathbf{\Omega} \times$ using the hat map

$$\widehat{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} , \quad \widehat{\Omega}\mathbf{v} = \mathbf{\Omega} \times \mathbf{v} , \quad [\widehat{a}, \widehat{b}]^\vee = \mathbf{a} \times \mathbf{b}$$

- Define reduced Lagrangian $\ell = \ell(\mathbf{\Omega})$, and variations $\mathbf{\Sigma} = (\Lambda^T \delta\Lambda)^\vee$

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- Define reduced Lagrangian $\ell = \ell(\boldsymbol{\Omega})$, and variations $\boldsymbol{\Sigma} = (\Lambda^T \delta\Lambda)^\vee$
- A short calculation gives

$$\begin{cases} \dfrac{d}{dt}\widehat{\Sigma} = \dfrac{d}{dt}\left(\Lambda^{-1}\delta\Lambda\right) = -\widehat{\Omega}\widehat{\Sigma} + \Lambda^{-1}\delta\dot{\Lambda} \\ \delta\widehat{\Omega} = \delta\left(\Lambda^{-1}\dot{\Lambda}\right) = -\widehat{\Sigma}\widehat{\Omega} + \Lambda^{-1}\delta\dot{\Lambda} \end{cases} \Rightarrow \begin{cases} \delta\Omega = \dot{\Sigma} + [\widehat{\Omega}, \widehat{\Sigma}] \quad \text{or} \\ \delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Sigma}} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma} \end{cases}$$

- Critical action principle is then given by

$$\delta \int \ell(\boldsymbol{\Omega}) \mathrm{d}t = \int \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \delta\boldsymbol{\Omega} \mathrm{d}t = -\int \left(\frac{d}{dt}\frac{\partial \ell}{\partial \boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \frac{\partial \ell}{\partial \boldsymbol{\Omega}} \right) \cdot \boldsymbol{\Sigma} = 0$$

- When $\ell = \frac{1}{2}\mathbb{I}\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}$, we get exactly Euler's equations of motion.

Euler-Poincaré theory

The theory can be extended to an arbitrary Lie group

1. $SO(3) \to G$, an arbitrary Lie group, with its Lie algebra $\mathfrak{g}$.
2. Invariance $L(hg, h\dot{g}) = L(g, \dot{g})$ for all fixed $h \in G$
3. Cross-products $\to$ adjoint and co-adjoint actions of Lie group
4. Poisson brackets
5. Conservation laws
6. Variational methods allow to make progress for cases that are difficult (impossible) to analyze using force balance
7. Calculations are formal and are based on known geometric objects, *i.e.*, computing actions on Lie algebras *etc.*
8. Applications of variational methods to problems of increasing complexity, for cases that are difficult to solve otherwise

Exact geometric rod theory, 1

- Independent variables: time t and material parameter s (not necessarily arc length)
- Dependent variables: position of the centerline $\mathbf{r}(s,t)$ and orientation $\Lambda(s,t)$, with $(\Lambda, \mathbf{r}) \in SE(3)$ (group of rotations and translations).
- Lagrangian (dots$= \partial_t$, primes $= \partial_s$) $\mathcal{L} = \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{r}', \Lambda, \dot{\Lambda}, \Lambda')$
- Use $SE(3)$ symmetry reduction [1] to reduce the Lagrangian to $\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma})$ of the following coordinate-invariant variables

$$\boldsymbol{\Gamma} = \Lambda^{-1}\mathbf{r}' \,, \quad \boldsymbol{\Omega} = \Lambda^{-1}\Lambda' \,, \text{(Darboux vector)} \tag{1}$$

$$\boldsymbol{\gamma} = \Lambda^{-1}\dot{\mathbf{r}} \,, \quad \boldsymbol{\omega} = \Lambda^{-1}\dot{\Lambda} \,. \text{(Angular velocity)} \tag{2}$$

- Note that symmetry reduction for elastic rods is *left-invariant* (reduces to body variables).
- Notation: small letters (e.g. $\boldsymbol{\omega}, \boldsymbol{\gamma}$) denote time derivatives; capital letters (e.g. $\boldsymbol{\Omega}, \boldsymbol{\Gamma}$) denote the s-derivatives.

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[1] Simo, Marsden, Krishnaprasad (1988) (SMK)

Exact geometric rod theory, 2

- Euler Poincaré theory [2] applied to elastic rods [3]: consider $\boldsymbol{\Sigma} = \Lambda^{-1}\delta\Lambda \in \mathfrak{so}(3)$ and $\boldsymbol{\Psi} = \Lambda^{-1}\delta\mathbf{r} \in \mathbb{R}^3$, and $(\boldsymbol{\Sigma}, \boldsymbol{\Psi}) \in \mathfrak{se}(3)$.

$$\delta\boldsymbol{\omega} = \frac{\partial\boldsymbol{\Sigma}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\gamma} = \frac{\partial\psi}{\partial t} + \boldsymbol{\gamma} \times \boldsymbol{\Sigma} + \boldsymbol{\omega} \times \psi \qquad (3)$$

$$\delta\boldsymbol{\Omega} = \frac{\partial\boldsymbol{\Sigma}}{\partial s} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\Gamma} = \frac{\partial\psi}{\partial s} + \boldsymbol{\Gamma} \times \boldsymbol{\Sigma} + \boldsymbol{\Omega} \times \psi, \qquad (4)$$

- Compatibility conditions (cross-derivatives in s and t are equal)

$$\boldsymbol{\Omega}_t - \boldsymbol{\omega}_s = \boldsymbol{\Omega} \times \boldsymbol{\omega}\,, \quad \boldsymbol{\Gamma}_t + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \boldsymbol{\gamma}_s + \boldsymbol{\Omega} \times \boldsymbol{\gamma}\,.$$

- Critical action principle $\delta \int \ell \mathrm{d}t\mathrm{d}s = 0+$ (3,4) give SMK equations.

$$0 = \delta \int \ell \mathrm{d}t\mathrm{d}s = \int \left\langle \frac{\delta\ell}{\delta\boldsymbol{\omega}}\,,\, \delta\boldsymbol{\omega} \right\rangle + \int \left\langle \frac{\delta\ell}{\delta\boldsymbol{\Omega}}\,,\, \delta\boldsymbol{\Omega} \right\rangle + \ldots$$

$$= \int \langle \text{linear momentum eq}, \boldsymbol{\Psi} \rangle + \langle \text{angular momentum eq}, \boldsymbol{\Sigma} \rangle \,\mathrm{d}t\mathrm{d}s$$

- These equations are equivalent to the Cosserat's equations for elastic rods (SMK1988, Ellis et al. 2009)

[2] See, e.g., Holm, Marsden, Ratiu 1998

[3] Ellis, Holm, Gay-Balmaz, VP and Ratiu, *Arch. Rat.Mech. Anal.*, (2010)

Elastic rods with non-local interactions

- Electrostatic interactions (e.g. polymers) or van-der-Vaals forces acting between different points in s, with the potential depending on Euclidian distance between the charges
- Static states can be computed by energy minimization [4]. Dynamics and geometry of the problem is computed in our work [5]
- We consider N elastic charges positioned, at each point s, at a vector $\boldsymbol{\eta}_m(s)$ in the local body frame.
- A charge k is at the position $\mathbf{c}_k(s,t) = \mathbf{r}(s,t) + \Lambda(s,t)\boldsymbol{\eta}_k(s)$, Euclidian distance $d_{k,m}(s,s')$ between charges at different points is

$$d_{k,m}(s,s') = |\mathbf{c}_m(s') - \mathbf{c}_k(s)|\, |\boldsymbol{\kappa}(s,s') + \xi(s,s')\boldsymbol{\eta}_m(s') - \boldsymbol{\eta}_k(s)| \text{ , where}$$

$$\boldsymbol{\kappa}(s,s') := \Lambda^{-1}(s)\left(\mathbf{r}(s') - \mathbf{r}(s)\right) \in \mathbb{R}^3 \quad \text{and} \quad \xi(s,s') := \Lambda^{-1}(s)\Lambda(s') \in SO(3)$$

- Thus, the total Lagrangian is $\ell = \ell_{\rm loc}(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}) + \ell_{\rm np}$, with

$$\ell_{\rm np}(\xi, \boldsymbol{\kappa}, \boldsymbol{\Gamma}) = \iint U\big(\xi(s,s'), \boldsymbol{\kappa}(s,s'), \boldsymbol{\Gamma}(s), \boldsymbol{\Gamma}(s')\big) \mathrm{d}s\mathrm{d}s'$$

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[4] See, e.g. Dichmann, Li, Maddocks, IMA Vol. Math Appl. 1996

[5] D. D. Holm, VP, CR Acad Sci. Paris, 2009; D. Ellis, F. Gay-Balmaz, D. D. Holm, VP, T. S. Ratiu, *ARMA*, 2010

Variations and equations of motion

- Critical action principle $\delta S = \delta \int \ell_{\rm loc}(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}) + \ell_{\rm np} \mathrm{d}t = 0$
- Linear momentum equation is the term multiplying $\boldsymbol{\Psi} = \Lambda^{-1}\delta\mathbf{r}$

$$\left(\frac{\partial}{\partial t} \frac{\delta l_{loc}}{\delta \boldsymbol{\gamma}} + \boldsymbol{\omega} \times \frac{\delta \ell_{loc}}{\delta \boldsymbol{\gamma}} \right) + \left(\frac{\partial}{\partial s} \frac{\delta \left(\ell_{loc} + \ell_{np} \right)}{\delta \boldsymbol{\Gamma}} + \boldsymbol{\Omega} \times \frac{\delta \left(\ell_{loc} + \ell_{np} \right)}{\delta \boldsymbol{\Gamma}} \right)$$
$$= \frac{\delta \ell_{loc}}{\delta \boldsymbol{\rho}} + \int \left(\xi(s,s') \frac{\partial U}{\partial \boldsymbol{\kappa}}(s',s) - \frac{\partial U}{\partial \boldsymbol{\kappa}}(s,s') \right) \mathrm{d}s' \,.$$

- Angular momentum equation is the term multiplying $\boldsymbol{\Sigma} = (\Lambda^{-1}\delta\Lambda)^{\vee}$

$$\left(\frac{\partial}{\partial t} \frac{\delta \ell_{loc}}{\delta \boldsymbol{\omega}} + \boldsymbol{\omega} \times \frac{\delta \ell_{loc}}{\delta \boldsymbol{\omega}} \right) + \left(\frac{\partial}{\partial s} \frac{\delta \ell_{loc}}{\delta \boldsymbol{\Omega}} + \boldsymbol{\Omega} \times \frac{\delta \ell_{loc}}{\delta \boldsymbol{\Omega}} \right) = \frac{\delta \ell_{loc}}{\delta \boldsymbol{\gamma}} \times \boldsymbol{\gamma}$$
$$+ \frac{\delta \left(\ell_{loc} + \ell_{np} \right)}{\delta \boldsymbol{\Gamma}} \times \boldsymbol{\Gamma} + \frac{\delta \ell_{loc}}{\delta \boldsymbol{\rho}} \times \boldsymbol{\rho} + \int \left(\frac{\partial U}{\partial \boldsymbol{\kappa}}(s,s') \times \boldsymbol{\kappa}(s,s') + \mathbf{Z}(s,s') \right) \mathrm{d}s' \,,$$

where the term $\boldsymbol{Z}(s,s')$ is the vector given by

$$\widehat{\boldsymbol{Z}}(s,s') = \xi(s,s') \left(\frac{\partial U}{\partial \xi}(s,s') \right)^T - \frac{\partial U}{\partial \xi}(s,s') \xi^T(s,s') \,.$$

- Steady helical states and stability can be computed analytically [6]

[6] S. Benoit, D. D. Holm and VP, J. Phys A, 2011

More complex example: a tube conveying fluid

Figure: Image of a garden hose and its mathematical description

- No friction in the system, incompressible fluid, Reynolds numbers $\sim 10^4$ (much higher in some applications), general 3D motions
- Hose can stretch and bend arbitrarily (inextensible also possible)
- Our work: arbitrary 3D motions, Lagrangians, Cross-section of the hose changes dynamically with deformations: *collapsible tube*, stretchable walls, compressible gas, . . .

Previous work

- *Constant fluid velocity in the tube*, 2D dynamics:
  English: Benjamin (1961); Gregory, Païdoussis (1966); Païdoussis (1998); Doare, De Langre (2002); Flores, Cros (2009), . . .
  Russian: Bolotin (?) (1956), Svetlitskii (monographs 1982, 1987), Danilin (2005), Zhermolenko (2008), Akulenko *et al.* (2015) . . .
  Hard to generalize to 3D, Not possible to consistently incorporate the cross-sectional dynamics
- Shell models: Paidoussis & Denise (1972), Matsuzaki & Fung (1977), Heil (1996), Heil & Pedley (1996) , . . . : Complex, computationally intensive, difficult (impossible) to perform analytic work for non-straight tubes.
- 3D dynamics from Cosserat's model: Bauergard, Goriely & Tabor (2010); Rivero-Rodriguez & Perez Saborid, (2015) Force balance, hard to accommodate dynamical change of the cross-section.
- Variational derivation: arbitrary Lagrangians, cross-sectional area change, 3D dynamics, initially curved pipes, . . . [7].

[7] FGB & VP (CR 2014, JNLS 2015 CR 2016), FGB, Georgievskii & VP, J., Fluids and Struct. 2018

Variational treatment: tubes with changing cross-sections

- Rod dynamics is described by $SE(3)$-valued functions (rotations and translations in space) $\pi(s,t) = (\Lambda, \mathbf{r})(s,t)$.
- Fluid dynamics inside the rod is described by 1D diffeomorphisms $s = \varphi(a,t)$, where a is the Lagrangian label.
- Conservation of 1-form volume element fluid is given by the constraint $Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = A|\boldsymbol{\Gamma}| = Q_0 \circ \varphi^{-1} \partial_s \varphi^{-1}$
- Incompressibility is enforced through Lagrange multiplier (pressure)
- Alternative way to write fluid conservation law is

$$\partial_t Q + \partial_s (Qu) = 0$$

  Note that commonly used $Au =$const does not conserve volume for time-dependent flow [8].
- The presence of fluid introduces a complication of right invariant part in a left invariant dynamics of the tube, which have fundamental consequences for *e.g.* conservation laws, dynamics etc.

[8] See *e.g.* Kudryashov *et al*, Nonlinear dynamics (2008) for correct derivation in 1D

Mathematics preliminaries: right-invariant incompressible fluid motion

- Following Arnold (1966), describe a 3D incompressible fluid motion by Diff_{Vol} group $\mathbf{r} = \varphi(\mathbf{a}, t)$.
- Eulerian fluid velocity is $\boldsymbol{u} = \varphi_t \circ \varphi^{-1}(\mathbf{r}, t)$; symmetry-reduced Lagrangian is $\ell = 1/2 \int |\boldsymbol{u}|^2 \mathrm{d}\mathbf{r}$.
- Variations of velocity are computed as

$$\boldsymbol{\eta} = \delta\boldsymbol{\varphi} \circ \boldsymbol{\varphi}^{-1}(\mathbf{a}, t)\,, \quad \delta\boldsymbol{u} = \boldsymbol{\eta}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\eta} - (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{u}\,. \quad (5)$$

- Incompressibility condition

$$J = \left|\frac{\partial \mathbf{r}}{\partial \mathbf{a}}\right| = 1 \Rightarrow \text{Lagrange multiplier } p\,. \quad (6)$$

- Euler equations: $\delta \int \ell \,\mathrm{d}V\,\mathrm{d}t = 0$ with (5) and (6)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p\,, \quad \mathrm{div}\,\boldsymbol{u} = 0$$

Garden hoses: Lagrangian and symmetry reductions

1. Symmetry group of the system (ignoring gravity for now)
$$G = SE(3) \times \mathrm{Diff}_A(\mathbb{R}) = SO(3) \circledS \mathbb{R} \times \mathrm{Diff}_A(\mathbb{R})\,. \qquad (7)$$
2. Position of elastic tube and fluid:
$$(\pi,\varphi)\cdot\Big(\big(\Lambda_0,\mathbf{r}_{t,0}\big),\mathbf{r}_f\Big) = \Big(\underbrace{\pi\cdot\big(\Lambda_0,\mathbf{r}_{t,0}\big)}_{\text{left invariant}},\underbrace{\pi\cdot\mathbf{r}_f\circ\varphi^{-1}(s,t)}_{\text{right invariant}}\Big)\,.$$
3. Velocities:
$$\begin{aligned}\big(\mathbf{v}_r\,,\,\mathbf{v}_f\big) =& \frac{d}{dt}\Big(\mathbf{r}(s,t)\,,\,\mathbf{r}\circ\varphi^{-1}(s,t)\Big)\\ =&\Big(\dot{\mathbf{r}}(s,t),\dot{\mathbf{r}}\circ\varphi^{-1}(s,t)+\mathbf{r}'(s,t)u(s,t)\Big)\,. \qquad (8)\end{aligned}$$
4. Change in cross-section $A = A(\boldsymbol{\Omega},\boldsymbol{\Gamma})$
5. Incompressibility condition $J = A(s,t)\frac{\partial a}{\partial s}|\boldsymbol{\Gamma}| = 1$ with Lagrange multiplier μ (pressure)
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s}(Qu) = 0\,, \quad \text{with} \quad Q = A|\boldsymbol{\Gamma}|\,. \qquad (9)$$

Equations of motion

Equations for arbitrary $\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u)$ and $A = A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$

$$\begin{cases} (\partial_t + \boldsymbol{\omega}\times)\dfrac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \dfrac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\delta \ell}{\delta \boldsymbol{\Omega}} - \dfrac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma} \times \left(\dfrac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = 0 \\ (\partial_t + \boldsymbol{\omega}\times)\dfrac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = 0 \\ m_t + \partial_s(mu - \mu) = 0, \quad m := \dfrac{1}{Q}\dfrac{\delta \ell}{\delta u} \\ \partial_t Q + \partial_s(Qu) = 0, \quad Q = A|\boldsymbol{\Gamma}| \\ \partial_t \boldsymbol{\Omega} = \boldsymbol{\omega} \times \boldsymbol{\Omega} + \partial_s \boldsymbol{\omega}\,, \qquad \partial_t \boldsymbol{\Gamma} + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \partial_s \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \boldsymbol{\gamma} \end{cases}$$

See our work for linear stability analysis, nonlinear traveling wave solutions *etc.* [9]

For constant cross-section, these equations are equivalent to Cosserat rod-based derivation [10]

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[9] F. Gay-Balmaz and VP, CR Acad Sci Paris (2014), JNLS (2015), FGB, D. Georgievskii and VP, J. Fluids and Struct. (2018)

[10] Bauergard, Goriely & Tabor, Int. J. Solids Struct. 2010; Rivero-Rodriguez & Perez Saborid, J. Fluids and Struct. 2015

Linear stability calculations for a finite tube

Our linear stability generalize earlier work [11] reducing to Timoshenko's beam vs Euler's beam in earlier models. Computational example:

- Soft rubber tube, Young's modulus $E = 10^7$ N/m^2.
- Diameter: 1cm, length: 1m, wall thickness: 2 mm.
- Area deformation parameter $A = A_0 - K|\mathbf{\Omega}|^2/2$, $K = 0.1R^2 = 2.5 \cdot 10^{-6}$ m^2.

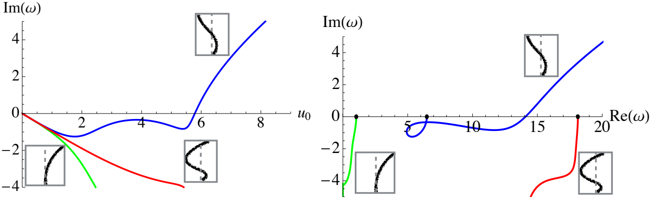


Figure: Left: $\mathrm{Im}(\omega)$ vs u (m/s), right: $\mathrm{Im}(\omega)$ vs $\mathrm{Re}(\omega)$. Instability of second mode for $u_0 \simeq 5.7$m/sec.

[11] Benjamin, JFM (1961), Gregory & Paidoussis Proc Roy Soc A (1966), see also book by Paidoussis (2004)

Effect of the area change in the tube

Results for instability $A = A_0 - K|\mathbf{\Omega}|^2/2$ for $K > 0$ (solid black) and $K = 0$ (red dashed).

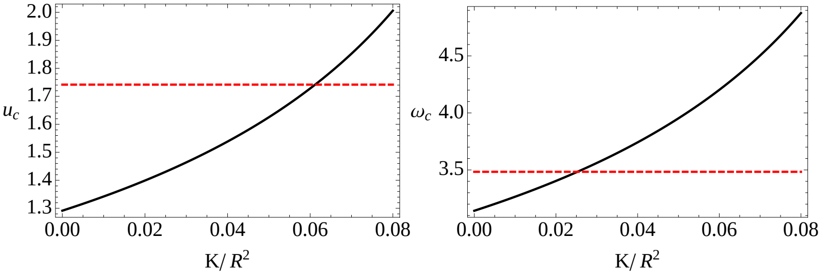


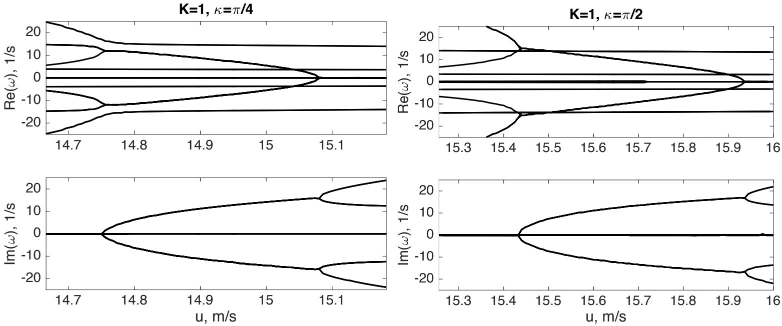
Figure: Left: critical velocity, m/s, Right: critical value of ω, in 1/s.

Additional results using geometric variational methods:

- Analytical solutions for nonlinear traveling waves
- Consideration of arbitrary nozzles at the exit
- Analytical treatment of linear stability analysis for helical geometry due to $SE(3)$ symmetry

Stability of initially helical tubes

- In the geometric-based derivation, linearization about a helical state reduces to a system with constant coefficients because of $SE(3)$ symmetry.
- Helix is parameterized by its Darboux vector $\boldsymbol{\Omega}_0 = K\pi(\cos\kappa, \sin\kappa, 0)^T/L$, with 2 parameters K and κ.
- Such results are very difficult to obtain in traditional approaches.



Flexible tubes with stretchable walls conveying fluid [12]

- Take, e.g. a circular cross-section, $A = \pi R^2$, and $R = R(s,t)$.
- Configuration manifold

$$\begin{aligned}\mathcal{Q} := & \ \mathcal{F}\left([0,L], SO(3)\times\mathbb{R}^3\times I_R\right) \\ & \times \left\{\varphi : \varphi^{-1}[0,L]\to[0,L] \mid \varphi \text{ diffeomorphism}\right\}.\end{aligned}$$

where I_R is the interval for allowed values of R, *e.g.* $I_R = \mathbb{R}_+$.
Blue=tube; Red=fluid.

- Lagrangian $\ell = \ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u, R, \dot{R}, R')$ (and, perhaps, R'' *etc*)
- Kinetic energy of the rod:

$$K_{\rm rod} = \frac{1}{2}\int_0^L \left(\alpha|\boldsymbol{\gamma}|^2 + a\dot{R}^2 + \mathbb{I}(R)\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)|\boldsymbol{\Gamma}|\mathrm{d}s,$$

- Eulerian velocity $u(t,s) = \left(\partial_t\varphi\circ\varphi^{-1}\right)(t,s), \quad s\in[0,L]$.
- Kinetic energy of the fluid:

$$K_{\rm fluid} = \frac{1}{2}\int_0^L (\xi_0\circ\varphi^{-1})\partial_s\varphi^{-1}\left|\boldsymbol{\gamma}+\boldsymbol{\Gamma}u\right|^2\mathrm{d}s,$$

[12] FGB & VP, *J. Nonlinear Sciences*, accepted (2018).

Compressible fluid case

1. No incompressibility condition anymore; pressure is not a Lagrange multiplier but a thermodynamic variable.
2. Introduce the effective line density of fluid $\xi(t,s) = \rho(t,s)Q(t,s)$, conservation law is
$$\xi(t,s) = \left[(\xi_0 \circ \varphi^{-1})\partial_s \varphi^{-1}\right](t,s)$$
3. Compressible fluid has additional internal energy density $e(\rho, S)$ with
$$E_{\rm int} = \int_0^L \xi e(\rho, S) \mathrm{d}s.$$
4. The pressure and temperature are defined by thermodynamic identities
$$\mathrm{d}e = -p\,\mathrm{d}\left(\frac{1}{\rho}\right) + T\mathrm{d}S \quad \Rightarrow$$
$$p(\rho, S) = \rho^2 \frac{\partial e}{\partial \rho}(\rho, S)\,, \quad T(\rho, S) = \frac{\partial e}{\partial S}(\rho, S)\,.$$

Equations of motion

1. Define the Lagrangian in reduced variables

$$\ell(\boldsymbol{\omega},\boldsymbol{\gamma},\boldsymbol{\Omega},\boldsymbol{\Gamma},u,\xi,S,R,\dot{R}) = \int_0^L \Big[\ell_0(\boldsymbol{\omega},\boldsymbol{\gamma},\boldsymbol{\Omega},\boldsymbol{\Gamma},u,\xi,R,\dot{R},R') - \xi e(\rho,S) - p_{\rm ext}Q\Big]\mathrm{d}s\,.$$

2. Variational principle (implicit dependence on R_s, R_{ss} *etc*).

$$\delta\int_0^T \ell(\boldsymbol{\omega},\boldsymbol{\gamma},\boldsymbol{\Omega},\boldsymbol{\Gamma},u,\xi,S,R,\dot{R})\mathrm{d}t = 0$$

on variations satisfying

$$\delta\boldsymbol{\omega} = \frac{\partial\boldsymbol{\Sigma}}{\partial t} + \boldsymbol{\omega}\times\boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\gamma} = \frac{\partial\psi}{\partial t} + \boldsymbol{\gamma}\times\boldsymbol{\Sigma} + \boldsymbol{\omega}\times\boldsymbol{\Psi}$$

$$\delta\boldsymbol{\Omega} = \frac{\partial\boldsymbol{\Sigma}}{\partial s} + \boldsymbol{\Omega}\times\boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\Gamma} = \frac{\partial\psi}{\partial s} + \boldsymbol{\Gamma}\times\boldsymbol{\Sigma} + \boldsymbol{\Omega}\times\boldsymbol{\Psi},$$

$$\delta u = \partial_t\eta + u\partial_s\eta - \eta\partial_s u$$

$$\delta\xi = -\partial_s(\xi\eta)\,, \quad \delta S = -\eta\partial_s S$$

Equations of motion (derivation)

1. Angular momentum: terms multiplying $\boldsymbol{\Sigma} = (\Lambda^{-1}\delta\Lambda)^{\vee}$
2. Linear momentum: terms multiplying $\boldsymbol{\Psi} = \Lambda^{-1}\delta\mathbf{r}$
3. Fluid meomentum: terms multiplying $\eta = \delta\varphi \circ \varphi^{-1}$
4. Wall momentum: Euler-Lagrange equations from δR
5. Advection equations for entropy from δS
6. Incompressible fluid equations can be obtained as well with pressure being Lagrange multiplier for incompressibility
7. Friction can be incorporated in our model using extra forces (Lagrange-d'Alembert's principle, neglected here)

Equations of motion (explicit)

Introducing for brevity

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{\omega}\times\,, \qquad \frac{D}{Ds} := \frac{\partial}{\partial s} + \boldsymbol{\Omega}\times$$

we obtain

$$\begin{cases} \dfrac{D}{Dt}\dfrac{\delta\ell}{\delta\boldsymbol{\omega}} + \boldsymbol{\gamma}\times\dfrac{\delta\ell}{\delta\boldsymbol{\gamma}} + \dfrac{D}{Ds}\dfrac{\delta\ell}{\delta\boldsymbol{\Omega}} + \boldsymbol{\Gamma}\times\dfrac{\delta\ell}{\delta\boldsymbol{\Gamma}} = 0 & \text{Rod, angular momentum} \\ \dfrac{D}{Dt}\dfrac{\delta\ell}{\delta\boldsymbol{\gamma}} + \dfrac{D}{Ds}\dfrac{\delta\ell}{\delta\boldsymbol{\Gamma}} = 0 & \text{Rod, linear momentum} \\ \partial_t\dfrac{\delta\ell}{\delta u} + u\partial_s\dfrac{\delta\ell}{\delta u} + 2\dfrac{\delta\ell}{\delta u}\partial_s u = \xi\partial_s\dfrac{\delta\ell}{\delta\xi} - \dfrac{\delta\ell}{\delta S}\partial_s S & \text{Fluid momentum} \\ \partial_t\dfrac{\delta\ell}{\delta\dot{R}} - \dfrac{\delta\ell}{\delta R} = 0 & \text{Rod, wall momentum (E-L eqs)} \\ \partial_t\boldsymbol{\Omega} = \boldsymbol{\Omega}\times\boldsymbol{\omega} + \partial_s\boldsymbol{\omega}, \quad \partial_t\boldsymbol{\Gamma} + \boldsymbol{\omega}\times\boldsymbol{\Gamma} = \partial_s\boldsymbol{\gamma} + \boldsymbol{\Omega}\times\boldsymbol{\gamma} & \text{Compatibility} \\ \partial_t\xi + \partial_s(\xi u) = 0, \quad \partial_t S + u\partial_s S = 0, & \text{Mass and entropy transport} \end{cases}$$

Solutions for shock waves

To derive shock conditions, (Rankine-Hugoniot), need to find equations of motion for fluid mass, momentum and energy in conservation form.

1. Mass: $\partial_t \xi + \partial_s(\xi u) = 0$
2. Fluid momentum 'conservation'

$$\partial_t\big(\xi\boldsymbol{\Gamma}\cdot(\boldsymbol{\gamma}+u\boldsymbol{\Gamma})\big)+\partial_s\big(u\xi\boldsymbol{\Gamma}\cdot(\boldsymbol{\gamma}+u\boldsymbol{\Gamma})+pQ\big)-\xi(\boldsymbol{\gamma}+u\boldsymbol{\Gamma})(\partial_s\boldsymbol{\gamma}+u\partial_s\boldsymbol{\Gamma}) = p\partial_s Q.$$

3. Energy conservation : Define the linear energy density E as

$$\mathbb{E} = \int_0^L E \mathrm{d}s\,, \qquad E := \xi e + \dot{R}\frac{\partial \ell_0}{\partial \dot{R}} + \boldsymbol{\omega}\cdot\frac{\partial \ell_0}{\partial \boldsymbol{\omega}} + \boldsymbol{\gamma}\cdot\frac{\partial \ell_0}{\partial \boldsymbol{\gamma}} + u\frac{\partial \ell_0}{\partial u} - \ell_0\,.$$

Then, $\boxed{\partial_t E + \partial_s J = 0}$ for the energy flux J given by

$$J := \boldsymbol{\omega}\cdot\frac{\partial \ell_0}{\partial \boldsymbol{\Omega}} + \boldsymbol{\gamma}\cdot\frac{\partial \ell_0}{\partial \boldsymbol{\Gamma}} + \dot{R}\frac{\partial \ell_0}{\partial R'} + u^2\frac{\partial \ell_0}{\partial u} - \xi u\frac{\partial \ell_0}{\partial \xi} + p\boldsymbol{\gamma}\cdot\frac{\partial Q}{\partial \boldsymbol{\Gamma}} + \left(\frac{p}{\rho}+e\right)\xi u.$$

These conservation laws are valid for arbitrary configurations of the tube – not necessarily a straight line.

Rankine-Hugoniot conditions across shock waves

Integrate the conservation laws for mass, momentum and energy across the shock

$$c[\rho] = [\rho u]$$

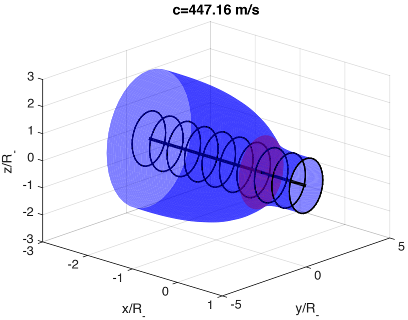
$$c\,[\rho]\,\boldsymbol{\Gamma}\cdot\boldsymbol{\gamma} + c\,[\rho u]\,|\boldsymbol{\Gamma}|^2 = [\rho u]\,\boldsymbol{\Gamma}\cdot\boldsymbol{\gamma} + \left[\rho u^2\right]\,|\boldsymbol{\Gamma}|^2 + [p]$$

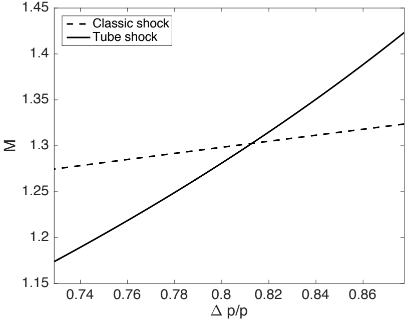
$$c\left[\rho\left(e + \frac{1}{2}\,|\boldsymbol{\gamma} + \boldsymbol{\Gamma} u|^2\right)\right] = \left[\frac{1}{2}\rho u\,|\boldsymbol{\gamma} + \boldsymbol{\Gamma} u|^2 + \frac{p}{|\boldsymbol{\Gamma}|^2}\boldsymbol{\Gamma}\cdot(\boldsymbol{\gamma} + \boldsymbol{\Gamma} u) + \rho u e\right]$$

1. These shock conditions are valid for arbitrary configuration of the tube
2. They reduce for standard 1D conditions when the tube is straight and non-deformable and motion of gas is in one direction only
3. The conditions satisfy the condition of entropy jump across the shock $[S] \geq 0$, so the shock is an irreversible process.
4. Impossible to guess these conditions by other methods

An example of the shock wave solution

Shock wave is propagating in a tube with thin rubber walls similar to the walls of a latex balloon.





Left: an example of a shock wave propagating in the tube. Right: Mach number of the shock as a function of shock's strength.

Comparison with previous works

1. Describes 1D gas motion in channel with variable cross-section (Witham 1974)
2. Equations for incompressible fluid case reduce to the models taking into account wall's inertia [13] (G =flux)

$$\begin{cases} \partial_t G + \partial_s \left(\alpha \frac{G^2}{A} \right) + \frac{A}{\rho} \partial_s p + K \frac{G}{A} = 0 \,, \quad \partial_t A + \partial_s G = 0, \\ \alpha \frac{\partial^2 R}{\partial t^2} - \gamma_1 \frac{\partial R}{\partial t} - a \frac{\partial^2 R}{\partial s^2} - c \frac{\partial^3 R}{\partial s^2 \partial t} + bR = p - p_{\rm ext} \end{cases}$$

3. Equations for incompressible fluid with stretchable walls coincides exactly with the models of arterial blood flow in a straight tube [14]

$$\begin{cases} \rho_0 (\partial_t u + u \partial_s u) = -\partial_s p - \tau(u, A) \\ \partial_t A + \partial_s (Au) = 0 \,, \quad p - p_{\rm ext} = \Phi(A) - T \partial_{ss} A, \end{cases}$$

4. Describes pressure wave propagation along the tube (pulse)

[13] Quarteroni, Tuveri, Veneziani, Comput. Vis. Sci 2000, Formaggia, Lamponi, Quarteroni, J Engr. Math, 2003

[14] Pedley & Luo Theor. Comp. Fluid Dyn 1998, Tang et al, Trans. ASME, 2009, T. Secomb Rev. Physiol. 2018

Geometric representation and Poisson bracket

1. Let us ignore the stretchable wall for now; it will yield canonical part of Poisson bracket, and consider only the left- and right- invariant parts
2. Consider two Lie groups G and H, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. G acts on the left on a manifold P and that H acts on the right on a manifold N:

$$
\begin{aligned}
\textit{Left}: &\Phi : G \times P \to P, \quad (g,p) \mapsto \Phi_g(p), \quad \Phi_g \circ \Phi_h = \Phi_{gh} \\
\textit{Right}: &\Psi : H \times N \to N, \quad (h,n) \mapsto \Psi_h(n), \quad \Psi_g \circ \Psi_h = \Psi_{hg}.
\end{aligned}
$$

3. For expandable tube, $g(t) = (\Lambda(t), \mathbf{r}(t))$ and $h(t) = \varphi(t)$ (dependence on s is implied); with $P = (\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ and $N = (\xi, S)$
4. Corresponding group actions on P and N:

$$
\begin{aligned}
&(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) \mapsto \mathrm{Ad}_{(\Lambda,\mathbf{r})}(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) + (\Lambda, \mathbf{r})\partial_s(\Lambda, \mathbf{r})^{-1}, \\
&\xi \mapsto (\xi \circ \varphi)\partial_s \varphi \quad \text{and} \quad S \mapsto S \circ \varphi.
\end{aligned}
$$

Momentum maps

1. Given the Lie algebra elements $\zeta \in \mathfrak{g}$ and $u \in \mathfrak{h}$, the associated infinitesimal generators ζ_P and u_N, (v.f. on P, resp., N) are defined by

$$\zeta_P(p) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_{\exp(\varepsilon\zeta)}(p), \quad \text{resp.,} \quad u_N(n) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi_{\exp(\varepsilon u)}(n),$$

2. Cotangent lift momentum maps for actions of Lie groups

$$\mathbb{J}_L : T^*P \to \mathfrak{g}^*, \quad \langle \mathbb{J}_L(\alpha_p), \zeta \rangle = \langle \alpha_p, \zeta_P(p) \rangle$$
$$\mathbb{J}_R : T^*N \to \mathfrak{h}^*, \quad \langle \mathbb{J}_R(\alpha_n), u \rangle = \langle \alpha_n, u_N(n) \rangle$$

3. Reduced variables in Lagrangian

$$\text{Left-invariant}: \quad \zeta(t) = g(t)^{-1}\dot{g}(t) \in \mathfrak{g} \qquad p(t) = \Phi_{g(t)^{-1}}(p_0) \in P\,,$$
$$\text{Right-invariant}: \quad u(t) = \dot{h}(t)h(t)^{-1} \in \mathfrak{h} \qquad n(t) = \Psi_{h(t)^{-1}}(n_0) \in N.$$

Lie algebra elements and variations

1. Critical action principle $\delta \int_0^T \ell\big(\zeta(t), u(t), p(t), n(t)\big)\mathrm{d}t = 0$, on variations satisfying

$$\begin{aligned} &\textit{Left}: \quad \delta\zeta = \dot\sigma + [\zeta,\sigma] \qquad \delta p = -\sigma_P(p) \\ &\textit{Right}: \quad \delta u = \dot v - [u,v] \qquad \delta n = -v_N(n), \end{aligned}$$

2. Equations of motion

$$\begin{cases} \dfrac{d}{dt}\dfrac{\delta\ell}{\delta\zeta} - \mathrm{ad}^*_\zeta \dfrac{\delta\ell}{\delta\zeta} + \mathbb{J}_L\left(\dfrac{\delta\ell}{\delta p}\right) = 0 \\ \dfrac{d}{dt}\dfrac{\delta\ell}{\delta u} + \mathrm{ad}^*_u \dfrac{\delta\ell}{\delta u} + \mathbb{J}_R\left(\dfrac{\delta\ell}{\delta n}\right) = 0 \\ \dot p + \zeta_P(p) = 0, \quad \dot n + u_N(n) = 0 \end{cases}$$

coupled with Euler-Lagrange equations for radius, or some other set of parameters a (e.g. for elliptical or more complex profiles)

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\dot a} - \frac{\delta\ell}{\delta a} = 0\,.$$

Hamiltonian structure

1. Define variables for reduced Hamiltonian as $(g(t), \alpha(t), h(t), \beta(t)) \in T^*(G \times H)$

$$\mu(t) = g(t)^{-1}\alpha(t) \in \mathfrak{g}^* \qquad p(t) = \Phi_{g(t)^{-1}}(p_0) \in P$$

$$\nu(t) = \beta(t)h(t)^{-1} \in \mathfrak{h}^* \qquad n(t) = \Psi_{h(t)^{-1}}(n_0) \in N.$$

2. Then, Poisson bracket is given by

$$\dot{f} = \{f, h\}_L + \{f, h\}_R + \{f, h\}_{\text{Canonical}}, \quad \text{where}$$

$$\{f, h\}_L = -\left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right]\right\rangle + \left\langle \frac{\delta f}{\delta \mu}, \mathbb{J}_L\left(\frac{\delta h}{\delta p}\right)\right\rangle - \left\langle \frac{\delta h}{\delta \mu}, \mathbb{J}_L\left(\frac{\delta f}{\delta p}\right)\right\rangle$$

$$\{f, h\}_R = +\left\langle \nu, \left[\frac{\delta f}{\delta \nu}, \frac{\delta h}{\delta \nu}\right]\right\rangle + \left\langle \frac{\delta f}{\delta \nu}, \mathbb{J}_R\left(\frac{\delta h}{\delta n}\right)\right\rangle - \left\langle \frac{\delta h}{\delta \nu}, \mathbb{J}_R\left(\frac{\delta f}{\delta n}\right)\right\rangle.$$

3. For expandable tube with compressible fluid,

$$h(\boldsymbol{\pi}, \boldsymbol{\mu}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, \nu, \xi, S) = \int_0^L (\boldsymbol{\pi} \cdot \boldsymbol{\omega} + \boldsymbol{\mu} \cdot \boldsymbol{\gamma} + \nu u) \mathrm{d}s - \ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u, \xi, S),$$

$$\boldsymbol{\pi} = \frac{\delta \ell}{\delta \boldsymbol{\omega}}, \quad \boldsymbol{\mu} = \frac{\delta \ell}{\delta \boldsymbol{\gamma}}, \quad \nu = \frac{\delta \ell}{\delta u}, \quad \boldsymbol{\omega} = \frac{\delta h}{\delta \boldsymbol{\pi}}, \quad \boldsymbol{\gamma} = \frac{\delta h}{\delta \boldsymbol{\mu}}, \quad u = \frac{\delta h}{\delta \nu}.$$

Explicit expression for Poisson brackets

$$
\begin{aligned}
\{f,g\}_L = &-\int_0^L \boldsymbol{\pi}\cdot\left(\frac{\delta f}{\delta\boldsymbol{\pi}}\times\frac{\delta g}{\delta\boldsymbol{\pi}}\right)\mathrm{d}s-\int_0^L \boldsymbol{\mu}\cdot\left(\frac{\delta f}{\delta\boldsymbol{\mu}}\times\frac{\delta g}{\delta\boldsymbol{\pi}}-\frac{\delta g}{\delta\boldsymbol{\mu}}\times\frac{\delta f}{\delta\boldsymbol{\pi}}\right)\mathrm{d}s\\
&-\int_0^L \boldsymbol{\Omega}\cdot\left(\frac{\delta f}{\delta\boldsymbol{\Omega}}\times\frac{\delta g}{\delta\boldsymbol{\pi}}-\frac{\delta g}{\delta\boldsymbol{\Omega}}\times\frac{\delta f}{\delta\boldsymbol{\pi}}\right)\mathrm{d}s\\
&+\int_0^L \left(\frac{\delta f}{\delta\boldsymbol{\Omega}}\cdot\partial_s\frac{\delta g}{\delta\boldsymbol{\pi}}-\frac{\delta g}{\delta\boldsymbol{\Omega}}\cdot\partial_s\frac{\delta f}{\delta\boldsymbol{\pi}}\right)\mathrm{d}s\\
&-\int_0^L \boldsymbol{\Omega}\cdot\left(\frac{\delta f}{\delta\boldsymbol{\Gamma}}\times\frac{\delta g}{\delta\boldsymbol{\mu}}-\frac{\delta g}{\delta\boldsymbol{\Gamma}}\times\frac{\delta f}{\delta\boldsymbol{\mu}}\right)\mathrm{d}s\\
&-\int_0^L \boldsymbol{\Gamma}\cdot\left(\frac{\delta f}{\delta\boldsymbol{\Gamma}}\times\frac{\delta g}{\delta\boldsymbol{\pi}}-\frac{\delta g}{\delta\boldsymbol{\Gamma}}\times\frac{\delta f}{\delta\boldsymbol{\pi}}\right)\mathrm{d}s+\int_0^L \left(\frac{\delta f}{\delta\boldsymbol{\Gamma}}\cdot\partial_s\frac{\delta g}{\delta\boldsymbol{\mu}}-\frac{\delta g}{\delta\boldsymbol{\Gamma}}\cdot\partial_s\frac{\delta f}{\delta\boldsymbol{\mu}}\right)\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
\{f,g\}_R = &\int_0^L \nu\left(\frac{\partial g}{\partial\nu}\partial_s\frac{\partial f}{\partial\nu}-\frac{\partial f}{\partial\nu}\partial_s\frac{\partial g}{\partial\nu}\right)\mathrm{d}s+\int_0^L \xi\left(\frac{\partial g}{\partial\nu}\partial_s\frac{\partial f}{\partial\xi}-\frac{\partial f}{\partial\nu}\partial_s\frac{\partial g}{\partial\xi}\right)\mathrm{d}s\\
&+\int_0^L S\partial_s\left(\frac{\partial f}{\partial S}\frac{\partial g}{\partial\nu}-\frac{\partial f}{\partial\nu}\frac{\partial g}{\partial S}\right)\mathrm{d}s.
\end{aligned}
$$

Conclusions and future work

1. Similar calculations can be done for incompressible fluid with thermal properties
2. Next steps: Porous media treated as an ensemble of expandable tubes with certain orientations (with F. Gay-Balmaz and T. Farkhutdinov)
3. Introduction of friction using recent variational theory of Gay-Balmaz and Yoshimura (nonlinear, nonholonomic constraint for entropy).
4. Experimental realization of the flow in a shock tube
5. Variational methods for a tube with a splitting channel (fork, bifurcation).

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Thank you!