Vortices on the Triaxial Ellipsoid

Special Lecture in honor of James Montaldi

Jair Koiller
Universidade Federal de Juiz de Fora

joint work with Adriano Regis and Cesar Castilho
Congrats, James!
Mexico City, 2016
Support

CAPES/CNPq/PVE11-2012
with Darryl Holm/Tudor Ratiu

CAPES/CNPq/PVE089-2013
with Richard Montgomery/Alain Albouy

UFJF/Visiting professorship
Juiz de Fora, “Manchester of Brazil”

Looking forward to collaborations, personal and institutional!
The kid with the ball is the world’s expert in rolling systems. Who is he?
Vortices are only a small part of James’ interests

Singularity theory, bifurcations
Integrability of Hamiltonian systems
Bifurcations of equilibria, periodic orbits
Celestial Mechanics, Nonholonomic systems
Group theory (Lie and finite)
Equivariant cohomology
Functional analysis/measure theory
Fractal maps
But vortices are highlighted in his web page!

http://www.maths.manchester.ac.uk/~jm/Vortices/

http://www.maths.manchester.ac.uk/~jm/wiki/Vortices
http://www.maths.manchester.ac.uk/~jm/Vortices/Applet/
http://www.maths.manchester.ac.uk/~jm/Vortices/Maple/
PICTURES!

“These figures and explanations supplement the paper with Laurent-Polz and Mark Roberts Stability of Point Vortices on the sphere”

http://www.maths.manchester.ac.uk/~jm/wiki/Vortices/Stability
http://www.maths.manchester.ac.uk/~jm/wiki/Vortices/Ring
http://www.maths.manchester.ac.uk/~jm/wiki/Vortices/RingPole
http://www.maths.manchester.ac.uk/~jm/wiki/Vortices/Staggered
http://www.maths.manchester.ac.uk/~jm/wiki/Vortices/Ring2poles

One appreciates here the depth of James’ work!!
Point vortices on a sphere of radius $R$

$$\dot{x}_i = \frac{1}{4\pi R} \sum_{j=1}^{N} \frac{\Gamma_j (x_j \times x_i)}{(R^2 - x_i \cdot x_j)}.$$ 

$$\sum_j \Gamma_j$$ does not need to vanish.

Uniform counter-vorticity in the background.

Bogomolov, Dynamics of vorticity at a sphere, 1977

https://link.springer.com/article/10.1007/BF01090320
James Montaldi’s on Vortices, so far
(the wizard on bifurcations of relative equilibria)

MR1811389, 2001:
Relative equilibria of point vortices on the sphere,
with C. Lim and M. Roberts
Physica D, 148 (1-2), 97-135
this paper opens the research program

MR2031280 2003:
Vortex Dynamics on a Cylinder,
with A. Soulière, T. Tokieda
SIAM J. Applied Dynamical Systems 2:3, 417-430

von Karman vortex street, prior to symmetry breaking
Point vortices on the sphere: stability of symmetric relative equilibria,
with F. Laurent-Polz, M. Roberts (T. Ratiu fest)
Journal of Geometric Mechanics 3 (4), 439-486

Energy Momentum: Symmetry adapted basis block diagonalizes the Hessian, provides nonlinear stability

2011: Dynamics of poles with position-dependent strengths and its optical analogues,
with T. Tokieda, Physica D 240: 20, 1565-1684
Special Issue on Fluid Dynamics

Complex vorticities, sources-sinks included, Snell’s law and refraction analogues
MR3110136, 2013:
Deformation of geometry and bifurcations of vortex rings, with T. Tokieda
Recent Trends in Dynamical Systems, 335-370

Stability of the Thomson heptagon without recourse to the Birkhoff normal form

MR3390464, 2014:
Point vortices on the hyperbolic plane, with C. Nava-Gaxiola, JMP 55, 102702
Mentorship: vortices!!!

www.maths.manchester.ac.uk/~jm/UGprojects.html

Motion of symmetric arrays of vortices

Vortices are points in a fluid where the fluid is spinning (like an idealized whirlpool). In an ideal fluid, one can study how the vortices interact without any reference to the background fluid, and over the past 100 or so years there has been much interest in studying this system, which is in fact a Hamiltonian system. In this project the student will discover why it is Hamiltonian, and how to use this fact to investigate the many symmetric configurations of vortices that can occur (see the applet on my website for examples).

http://www.maths.manchester.ac.uk/~jm/Vortices/Applet/VortexApplet.html
Plan of the talk

I. Vortices on closed surfaces, special case of genus zero

II. Genus zero, vortex pair: equilibria. linearization
   - Surfaces of revolution: pair of poles is stable (always)
   - With antipodal symmetry: an invariant submanifold.

III. Examples
   - Surfaces of revolution, made easy!
   - Double faced elliptical region.
   - Triaxial ellipsoid
PART I.

Vortices on closed surfaces,
special case of genus zero

Conformal map of triaxial ellipsoid
to the sphere. Validation.
Motivation: Bose Einstein condensates

We nucleate pairs of vortices of opposite charge (vortex dipoles) by forcing superfluid flow around a repulsive Gaussian obstacle within the BEC. By controlling the flow velocity we determine the critical velocity for the nucleation of a single vortex dipole, with excellent agreement between experimental and numerical results. We present measurements of vortex dipole dynamics, finding that the vortex cores of opposite charge can exist for many seconds and that annihilation is inhibited in our trap geometry.
Single vortex on Bolza’s surface


http://dx.doi.org/10.1098/rspa.2017.0447
Has it a relation with Pascal Chossat’s H-planform?

Hyperbolic Planforms in Relation to Visual Edges and Textures Perception, Pascal Chossat, Olivier Faugeras, PLOS Computational Biology, 2009, https://doi.org/10.1371/journal.pcbi.1000625
Point vortices on a sphere of radius $R$

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Hally’s equations (1980)
Non euclidian metrics in the plane

Surface: $\Sigma \equiv \mathbb{C}$ with $ds^2 = h^2(z, \bar{z})|dz|^2$.

\[
\dot{z}_n = h^{-2}(z_n, \bar{z}_n) \left[ \sum_{k \neq n}^N -i \frac{\Gamma_k}{z_n - z_k} + i \Gamma_n \frac{\partial}{\partial z_n} \ln(h(z_n, \bar{z}_n)) \right], \quad n = 1, .., N;
\]

For a (closed genus zero surface) Hally made an observation that one could still use the stereographic projection from a sphere conformally representing the surface to $\mathbb{C} \cup \infty$, but the above equations are only valid if the sum of vorticities vanishes.

In fact, we found that when $\sum_{i=1}^{N} \Gamma_i \neq 0$ there is an extra term It is nonlocal: involves $\Delta^{-1} h$.

(More in the end of the talk, time permitting, on higher genus. The fundamental object is Green function of the Laplace-Beltrami operator.)
References


Vortices on Closed Surfaces, with S. Boatto

Vortex pairs on genus zero surfaces \( \Sigma \)

We use \( S^2 \) to represent the dynamics.

\[
H = -\frac{1}{2} \ln \left( h(s_1)h(s_2)|s_1 - s_2|^2 \right) =
\]
\[
= -\left( \ln |s_1 - s_2| + \frac{1}{2} \ln h(s_1) + \frac{1}{2} \ln h(s_2) \right)
\]

\[
\Omega_{\text{pair}} = h^2(s_1)\Omega_o(s_1) - h^2(s_2)\Omega_o(s_2)
\]

where \( | \cdot | \) is the euclidian distance in \( \mathbb{R}^3 \), and \( \Omega_o \) = area form of the sphere.
Equations of motion

\[ \dot{s}_1 = \frac{1}{h^2(s_1)} \left( \frac{s_1 \times s_2}{|s_1 - s_2|^2} - \frac{1}{2}s_1 \times \text{grad} h(s_1)/h(s_1) \right) \]

\[ \dot{s}_2 = \frac{1}{h^2(s_2)} \left( \frac{s_1 \times s_2}{|s_1 - s_2|^2} + \frac{1}{2}s_2 \times \text{grad} h(s_2)/h(s_2) \right) . \]

We “only” need a conformal map from the surface \( \Sigma \) to \( S^2 \)
Basic facts

• Invariance of the ODEs under an arbitrary (6 parameter) Moebius transformation

\[ m : S^2 \rightarrow S^2. \]

• Any equilibrium pair can be made antipodal under a suitable \( m \).

• For surfaces with antipodal symmetry, there is a center manifold!
Point i) means that the system of EDOs is well defined. Two conformal maps from $\Sigma$ to $S^2$ (equivalently, to the extended complex plane $\mathbb{C} \cup \infty$) differ by a Moebius transformation. A Moebius transformation corresponds to a $SO(3)$ rotation of a suitably located dilated sphere. (We neglect linear time scale changes)

Point ii): although $\Sigma$ in general has no symmetries, in the representing phase space $S^2 \times S^2$, an antipodal pair $(s, -s)$ is an equilibrium if the gradient of the conformal factor vanishes at both $s$ and $-s$. Moreover, under a suitable Moebius transformation, any equilibrium can be represented as an antipodal pair.
Center manifold for surfaces with antipodal symmetry

Claim: the antipodal pairs $S_{\text{ant}} = \{(s_1, s_2) \in S^2 \times S^2 \mid s_2 = -s_1\}$ for an invariant submanifold.

$$H_{\text{ant}} = -\frac{1}{2} \ln h(s), \quad \Omega_{\text{ant}} = h^2(s)\Omega_0(s)$$

$$\dot{s} = -\frac{1}{2} s \times \text{grad} h(s) / h^3(s)$$

Proof. Pick any point $(s_1, s_2)$, with $s_2 = -s_1$.

The first terms in the ODES disappear. The second equation becomes

$$s_2' = \frac{1}{h^2(s_2)} \left(\frac{1}{2} s_2 \times \text{grad} h(s_2) / h(s_2)\right).$$

$$= \frac{1}{h^2(s_1)} \left(\frac{1}{2} (-s_1) \times [-\text{grad} h(s_1) / h(s_1)]\right) = -s_1'$$

that reproduces the first equation.
How to get a conformal map from the surface $\Sigma$ to $S^2$?

- Analytically, uniformization is related to Laplace Beltrami operator, Ricci flow, etc...

- Numerical methods are well developed

  M. Desbrun (Caltech),

  X. Gu (Stony Brook, a S. T. Yau student),

  computer graphic community

- Combine with symplectic integrators in the sphere
A simple example that one can do analytically: Matrioschka

The conformal map is simple to find.

One can show: the pair of vortices at the poles is in stable equilibrium and give the frequencies

Left as homework
Conformal map of the triaxial ellipsoid

\[ h^2 = \frac{(\lambda_2 - \lambda_1)}{(\mu_2 - \mu_1)} \]

\[ \mathbb{E}^2 : x^2/a + y^2/b + z^2/c = 1, \ a < b < c \]

\[ ds^2 = \frac{\lambda_2 - \lambda_1}{4} \left[ \frac{\lambda_1 d\lambda_1^2}{(\lambda_1 - a)(\lambda_1 - b)(\lambda_1 - c)} + \frac{-\lambda_2 d\lambda_2^2}{(\lambda_2 - a)(\lambda_2 - b)(\lambda_2 - c)} \right] \]

\[ S^2 \ (\text{with round metric}) \]

\[ ds^2 = \frac{\mu_2 - \mu_1}{4} \left[ \frac{d\mu_1^2}{(\mu_1 - I_1)(\mu_1 - I_2)(\mu_1 - I_3)} - \frac{d\mu_2^2}{(\mu_2 - I_1)(\mu_2 - I_2)(\mu_2 - I_3)} \right]. \]

I give a chocolate for prize

How to get the \( \lambda \)'s in terms of \( \mu \)'s? There is no need to solve Beltrami’s equation.

**Suggestion:** there should be a special choice of \( I_1, I_2, I_3 \) that is coherent with \( a, b, c \).

**Quiz (give at the end of the talk):** show that the conformal factor \( h^2 = \frac{\lambda_2 - \lambda_1}{\mu_2 - \mu_1} \) at the umbilical points is given by

\[
\frac{(b - a)(c - b)}{b(I_2 - I_1)(I_3 - I_2)}.
\]

The trick for this allows to compute derivatives of all orders of the conformal factor at the 10 special points (the 4 umbilics and 6 axis endpoints).
What are the uniformizing coordinates \((u, v)\)?
Elliptic functions to define common uniformizing coordinates: \((u, v)\) on an octant

‘Master Equation’

\[
\frac{K(k')}{K(k)} = r
\]

\[k^2 = \frac{I_2 - I_1}{I_3 - I_1}, \quad (k')^2 = \frac{I_3 - I_2}{I_3 - I_1}\]

\[r = r(a, b, c)\] function of the three ellipsoid axes

• Ellipsoid: elliptic integrals of third kind

• Sphere: elliptic integrals of the first kind

(details in the end if desired and time permitting)
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Global map from $\mathbb{E}^2$ to $\mathbb{S}^2$

A common orthogonal lattice in the $w = u + iv$ plane yields ramified double coverings for both surfaces.

Identify the corresponding points on the surfaces.

For global isothermal coordinates $(z, \bar{z})$ on $\mathbb{E}^2$, just compose with stereographic projection from the sphere to the complex $z$-plane:

$$w = u + iv \text{ (lattice)} \to (p \in \mathbb{E}^2) \leftrightarrow (\gamma \in \mathbb{S}^2) \to z \in \mathbb{C}.$$

$z = z(w)$ is a complex elliptic function.
https://en.wikipedia.org/wiki/Peirce_quincuncial_projection
Historical note: Felix Klein

All closed surfaces which can be conformally represented upon each other by means of a uniform correspondence, are, of course, to be regarded as equivalent for our purposes. For every complex function of position on the one surface will be changed by this representation into a similar function on the other surface; hence, the analytical relation which is graphically expressed by the co-existence of two complex functions on the one surface is entirely unaffected by the transition to the other surface. For instance, the ellipsoid may be conformally represented, by virtue of known investigations, on a sphere, in such a way that each point of the former corresponds to one and only one point of the latter; this shows us that the ellipsoid is as suitable for the representation of rational functions and their integrals as the sphere.

Felix Klein, On Riemann's Theory Of Algebraic Functions
http://www.gutenberg.org/ebooks/36959
Historical note: Riemann

Riemann began his study of the quotient $\frac{K'}{K}$ by determining how $K$ and $K'$, when regarded as functions of $k^2$, would change as $k^{-2}$ made a circuit around the branch points $0, 1, \infty$. For this purpose, he took the hypergeometric functions $K$ and $K'$ to be defined by the integrals

$$K = \int_0^1 (1-t^2)^{-\frac{1}{2}} (1-k^2t^2)^{-\frac{1}{2}} \, dt = \int_0^1 \frac{1}{2} x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} (1-k^2x)^{-\frac{1}{2}} \, dx \quad (7.75)$$

and

$$K' = \int_0^1 (1-t^2)^{-\frac{1}{2}} (1-k^2t^2)^{-\frac{1}{2}} \, dt = -i \int_1^{k^{-2}} \frac{1}{2} x^{-\frac{1}{2}} (1-x)^{-\frac{1}{2}} (1-k^2x)^{-\frac{1}{2}} \, dx. \quad (7.76)$$
We see from (7.74) that if we set $\tau = \frac{iK'}{K}$, then the change in $\frac{K'}{K}$ referred to by Jacobi may be written $\tau' = \frac{b + b' \tau}{a + a' \tau}$. Thus, as early as 1828, Jacobi was aware that $k^2$ was a modular function of $\frac{iK'}{K}$ with respect to the subgroup $\Gamma(2)$ of the full modular group. In his lectures on hypergeometric functions, Riemann proved Jacobi’s theorem that $k^2$ was invariant with respect to $\Gamma(2)$. In the course of the proof, he found the generators of the group $\Gamma(2)$ and constructed its fundamental domain. He also considered the inverse case, $\frac{K'}{K}$ as a function of $k^2$; he demonstrated that this function was a conformal mapping from the upper half-plane to half of the fundamental domain of $\Gamma(2)$. Note
Historical note: Jacobi

27. *Jacobi, suite des notices sur les fonctions elliptiques.*

27.

Suite des notices sur les fonctions elliptiques. (V. p. 192.)

(Par Mr. C. G. J. Jacobi, prof. en phil. à Königsberg.)

(Extrait d'une lettre de l'auteur au redacteur de ce journal, du 21. Juillet 1826.)

J'ajoute aux formules données dans ma dernière lettre les développements des fonctions elliptiques de seconde et de troisième espèce.

Soit en adoptant la notation de Mr. Legendre, $\triangle = \sqrt{1 - k^2 \sin \phi}$,

$E \phi = \int_0^\phi \triangle \, d \phi$, $E^1 = \int_0^{\pi/2} \triangle \, d \phi$, $F \phi = \int_0^\phi \frac{d \phi}{\triangle}$, $F^1 = \int_0^{\pi/2} \frac{d \phi}{\triangle}$, de manière que $F^1$ est ce que je désigne par $K$: si l'on mèt

$\phi = a m \frac{2Kx}{K}$, $a = \frac{e^{-K/\pi}}{K}$. 

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Les fonctions elliptiques diffèrent essentiellement des transcendantes ordinaires. Elles ont une manière d'être pour ainsi dire absolue. Leur caractère principal est d'embrasser tout ce qu'il y a de périodique dans l'analyse. En effet les fonctions trigonométriques ayant une période réelle,

Crelle's Journal. III. Bd. 3. Hft. 40

310 27. Jacobi, suite des notices sur les fonctions elliptiques.

les exponentielles une période imaginaire, les fonctions elliptiques embrassent les deux cas, puisqu'on a en même temps

\[ \sin am(u + 4K) = \sin am(u) \]

\[ \sin am(u + 2iK') = \sin am(u), \]

\( i \) étant \(-1\). D'ailleurs on démontre aisément qu'une fonction analytique ne saura avoir plus que deux périodes, l'une réelle et l'autre imaginaire ou l'une et l'autre imaginaires. Ce dernier cas répond à un module \( k \) imaginaire. Le quotient \( \frac{K'}{K} \) des deux périodes d'une fonction proposée détermine le module \( k \) des fonctions elliptiques par lesquelles elle doit être exprimée au moyen des formules 15., 17. Il conviendra peut-être à introduire dans l'analyse des fonctions elliptiques ce quotient \( \frac{K'}{K} \) comme module au lieu de \( k \). À l'égard de ce quotient j'ai trouvé
"que $k$ ne change pas de valeur, si l'on écrit au lieu de $\frac{K'}{K}$ l'expression
\[
\frac{bK + ib'\overline{K'}}{aK + ia'\overline{K'}} = \frac{KK' - i(a\overline{b}KK + a'b'\overline{K'K'})}{a\overline{a}KK + a'a'\overline{K'K'}}
\]
"$a$, $a'$, $b$, $b'$ étant des nombres entiers quelconques, $a$ un nombre
"impair, $b$ un nombre pair, tels que $ab' - a'b = 1$;"
théorème remarquable et qui doit être envisagé comme un des théo-
rèmes fondamentaux de l'analyse des fonctions elliptiques.

Les méthodes qui m'ont conduit à la théorie générale de la trans-
formation des fonctions elliptiques s'appliquent également à une classe très
étendue d'intégrales doubles, triples, et même d'intégrales multiples d'un
ordre quelconque. Un premier essai sur cette matière épineuse a été
donné dans un petit mémoire qui a pour titre:

"De singulari quadam duplicis integralis transformatione"
inséré au second volume de votre journal.

Vous voyez, Monsieur, que la théorie des fonctions elliptiques est
un vaste objet de recherches qui dans le cours de ses développements em-
brassent presque toute l'algèbre, la théorie des intégrales définies et la
science des nombres. Quel titre de gloire pour l'illustre auteur du traité
des fonctions elliptiques, que d'avoir créé cette belle théorie et d'avoir
allumé ce flambeau à la posterité.
Historical note:

Letter from Jacobi to Bessel
December 28 1838

“Ich habe vorgestern die geodätische Linie für ein Ellipsoid mit
drei ungleichen Achsen auf Quadraturen zurückgeführt. Es sind die
einfachsten Formeln von der Welt, Abelsche Integrale, die ich in
die bekannten elliptischen verwandeln, wenn man 2 Achsen gleich
setzt.”

The day before yesterday, I reduced to quadrature the
problem of geodesic lines on an ellipsoid with three un-
equal axes. They are the simplest formulas in the world,
Abelian integrals, which become the well known elliptic
integrals if 2 axes are set equal.”
Validation: test Kimura’s conjecture on the triaxial ellipsoid

Ellipsoid $a = 1$, $b = 6$, $c = 9$. 
Poincaré map. $a = 1, \ b = 4, \ c = 9, \ H = -60.$


http://aimsciences.org/article/doi/10.3934/jgm.2018007
VOXER PAIRS ON A TRIAXIAL ELLIPSOID
AND KIMURA’S CONJECTURE

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(Communicated by James Montaldi)

ABSTRACT. We consider the problem of point vortices moving on the surface of a triaxial ellipsoid. Following Hally’s approach, we obtain the equations of motion by constructing a conformal map from the ellipsoid into the sphere and composing with stereographic projection. We focus on the case of a pair of opposite vortices. Our approach is validated by testing a prediction by Kimura that a (infinitesimaly close) vortex dipole follows the geodesic flow. Poincaré sections suggest that the global flow is non-integrable.
Questions on part I?
Part II: Linearization at an antipodal equilibrium

(we will now run the slides fast, jumping to the characteristic polynomial, slide 43 and surfaces with antipodal symmetry, slide 52)
Linearization

Using a Moebius transformation we may assume that the antipodal pair is $s_{1,2}^* = (0, 0, \pm 1)$. Take the tangent planes for coordinates. Then we have a system of four nonlinear ODEs for $x_1, y_1, x_2, y_2$.

All we will need for the linearization are the quadratic expansions of the conformal factor at $s_{1,2}^* = (0, 0, \pm 1)$.

\[
H_1(x_1, y_1) = h_1 + \frac{1}{2} p_1 x_1^2 + \frac{1}{2} q_1 y_1^2
\]
\[
H_2(x_2, y_2) = h_2 + \frac{1}{2} p_2 (x_2')^2 + \frac{1}{2} q_2 (y_2')^2
\]

with $h_1, h_2 > 0$ and where

\[
x_2' = x_2 \cos \theta - y_2 \sin \theta , \quad y_2' = x_2 \sin \theta + y_2 \cos \theta .
\]
Are there restrictions on the parameters $h_i, p_i, q_i, \theta$?

One can create a Morse function on $S^2$ with arbitrary quadratic expansions at two chosen points.

But... there is an Embedding Problem:

Realize a surface $\Sigma \subset \mathbb{R}^3$ that is isometric to the ‘abstract’ Riemannian manifold $(S^2, h^2g_o)$.

Gromov showed that can be done in $\mathbb{R}^5$.

Finding sufficient conditions in $\mathbb{R}^3$ is an open question, even locally.

L. Nicolaescu, Counting Morse functions on the 2-sphere
doi:10.1112/S0010437X08003680
https://mathoverflow.net/questions/37708/nash-embedding-theorem-for-2d-manifolds
The linearized system

Substitute

\[ s_1 = (x_1, y_1, \sqrt{1-x_1^2-y_1^2}) \], \quad s_2 = (x_2, y_2, -\sqrt{1-x_2^2-y_2^2}) \]

in the nonlinear ODEs and retain only the linear terms:

\[
4h_1^2 (\dot{x}_1, \dot{y}_1) = (-y_1 - y_2, x_1 + x_2) + \frac{2}{h_1} (\partial H_1 / \partial y_1, -\partial H_1 / \partial x_1)
\]

\[
4h_2^2 (\dot{x}_2, \dot{y}_2) = (-y_1 - y_2, x_1 + x_2) + \frac{2}{h_2} (\partial H_2 / \partial y_2, -\partial H_2 / \partial x_2)
\]

**Quiz:** Why 4??
\[
\begin{align*}
\dot{s}_1 &= \frac{1}{h^2(s_1)} \left( \frac{s_1 \times s_2}{|s_1 - s_2|^2} - \frac{1}{2} s_1 \times \text{grad} \frac{h(s_1)}{h(s_1)} \right) \\
\dot{s}_2 &= \frac{1}{h^2(s_2)} \left( \frac{s_1 \times s_2}{|s_1 - s_2|^2} + \frac{1}{2} s_2 \times \text{grad} \frac{h(s_2)}{h(s_2)} \right).
\end{align*}
\]

Answer to quiz:

\[
|s_1 - s_2| = 2
\]

\[
s_1 \sim (x_1, y_1, +1), \ s_2 \sim (x_2, y_2, -1)
\]
Linearization Matrix

No loss of generality in assuming $h_1 = h_2 = h$.

$$4h^2 \dot{X} = AX, \quad X = (x_1, y_1, x_2, y_2)^t$$

The $4h^2$ in the left side is irrelevant (make a linear change of time).

$$A = \begin{bmatrix} 0 & \gamma_1 & 0 & -1 \\ \delta_1 & 0 & 1 & 0 \\ 0 & -1 & a & b \\ 1 & 0 & c & -a \end{bmatrix}$$
Characteristic polynomial (a biquadratic!)

\[ p(\lambda) = \lambda^4 - 2\rho \lambda^2 + \kappa \]

\[ \rho = \frac{1}{2} (\gamma_1 \delta_1 + \gamma_2 \delta_2) - 1 \quad , \quad \kappa = 1 + \gamma_1 \delta_1 \gamma_2 \delta_2 - \gamma_1 b - c \delta_1 . \]

Coefficients (depending on the data \( h, p_1, q_1, p_2, q_2, \theta \)):

\[
\begin{align*}
\gamma_1 &= -1 + 2q'_1, \quad q'_1 = q_1/h \\
\delta_1 &= 1 - 2p'_1, \quad p'_1 = p_1/h \\
\gamma_2 &= -1 + 2q'_2, \quad q'_2 = q_2/h \\
\delta_2 &= 1 - 2p'_2, \quad p'_2 = p_2/h \\
\end{align*}
\]

\[
\begin{align*}
a &= 2 \sin \theta \cos \theta \left[ q'_2 - p'_2 \right] \\
b &= -1 + 2 \left[ q'_2 \cos^2 \theta + p'_2 \sin^2 \theta \right] \\
c &= 1 - 2 \left[ p'_2 \cos^2 \theta + q'_2 \sin^2 \theta \right] \\
\end{align*}
\]

One verifies in the calculations a nice simplification: \( a^2 + bc = \gamma_2 \delta_2 \).
Details need to be jumped over ... 

Let’s go to surfaces with antipodal symmetry, slides 52-54
Claim: no loss of generality in assuming that $h_1 = h_2$.

Apply a Moebius transformation $g_\beta : S^2 \to S^2$ that fixes the poles, but shifts parallels up or down. It corresponds to a rescaling $r = \beta \tilde{r}$ in the complex plane $\mathbb{C}$. We call $\beta$ the adjustment parameter.

Calculation: we have spheres $(x, y, z)$ (old) and $(\tilde{x}, \tilde{y}, \tilde{z})$ (new):

$$z = \frac{\beta^2(1 + \tilde{z}) - (1 - \tilde{z})}{\beta^2(1 + \tilde{z}) + (1 - \tilde{z})}$$

Let $\phi, \tilde{\phi}$ denote the latitudes, measured from the equator. $z = \sin \phi$, $\tilde{z} = \sin \tilde{\phi}$ heights of the corresponding parallels.

Conformal factor of this map:

$$h_{old/new} = \frac{2\beta}{(\beta^2 - 1)\tilde{z} + (\beta^2 + 1)}$$
Substituting $\tilde{z} = \pm 1$ gives

$$h_{old/new}(-1) = \beta, \quad h_{old/new}(1) = 1/\beta$$

$$x \sim \beta \tilde{x}, \quad y \sim \beta \tilde{y} \text{ near south pole}$$

$$x \sim \frac{1}{\beta} \tilde{x}, \quad y \sim \frac{1}{\beta} \tilde{y} \text{ near north pole}$$

which in hindsight is what one should expect.

Since conformal factors multiply under composition, we can choose a suitable $\beta$ so that the conformal factors at the poles become equal (upon composition of the original map from $\Sigma$ to $S^2$, followed by $g_\alpha : S^2 (old) \to S^2 (new, tilded)$.)

The new common factor will be given by

$$h = h_1 \beta = h_2 / \beta = \sqrt{h_1 h_2}, \quad \text{with} \quad \beta = (h_2 / h_1)^{1/2}.$$
Transformed coefficients

\[ \frac{1}{2} - \frac{p_{1}^{\text{new}}}{h} = \beta^2 \left( \frac{1}{2} - \frac{p_1}{h_1} \right) \]

\[ \frac{1}{2} - \frac{q_{1}^{\text{new}}}{h} = \beta^2 \left( \frac{1}{2} - \frac{q_1}{h_1} \right) \]

\[ \frac{1}{2} - \frac{p_{2}^{\text{new}}}{h} = \left( \frac{1}{\beta^2} \right) \left( \frac{1}{2} - \frac{p_2}{h_2} \right) \]

\[ \frac{1}{2} - \frac{q_{2}^{\text{new}}}{h} = \left( \frac{1}{\beta^2} \right) \left( \frac{1}{2} - \frac{q_2}{h_2} \right) \]

where

\[ \beta^2 = \frac{h_2}{h_1}, \quad h = \sqrt{h_1 h_2}. \]

So we will assume \( h_1 = h_2 = h \) in the sequel.
Matrix and its characteristic polynomial

\[ 4h^2 \dot{X} = AX, \quad X = (x_1, y_1, x_2, y_2)^t \]

The \( 4h^2 \) in the left side is irrelevant (make a linear change of time).

\[
A = \begin{bmatrix}
0 & \gamma_1 & 0 & -1 \\
\delta_1 & 0 & 1 & 0 \\
0 & -1 & a & b \\
1 & 0 & c & -a
\end{bmatrix}
\]
Characteristic polynomial (a biquadratic!)

\[ p(\lambda) = \lambda^4 - 2\rho\lambda^2 + \kappa \]

\[ \rho = \frac{1}{2} (\gamma_1\delta_1 + \gamma_2\delta_2) - 1 , \ \kappa = 1 + \gamma_1\delta_1\gamma_2\delta_2 - \gamma_1b - c\delta_1 . \]

Coefficients (depending on the data \( h, p_1, q_1, p_2, q_2, \theta \)):

\[
\begin{align*}
\gamma_1 &= -1 + 2q'_1, & q'_1 &= q_1/h \\
\delta_1 &= 1 - 2p'_1, & p'_1 &= p_1/h \\
\gamma_2 &= -1 + 2q'_2, & q'_2 &= q_2/h \\
\delta_2 &= 1 - 2p'_2, & p'_2 &= p_2/h \\

a &= 2 \sin \theta \cos \theta [q'_2 - p'_2] \\
b &= -1 + 2 \left[ q'_2 \cos^2 \theta + p'_2 \sin^2 \theta \right] \\
c &= 1 - 2 \left[ p'_2 \cos^2 \theta + q'_2 \sin^2 \theta \right]
\end{align*}
\]

One verifies in the calculations a nice simplification: \( a^2 + bc = \gamma_2\delta_2 \).
When there is no twist ($\theta = 0$)

The characteristic polynomial is more symmetric

$$p(\lambda) = \lambda^4 + (2 - \delta_1 \gamma_1 - \delta_2 \gamma_2)\lambda^2 + (1 - \delta_1 \delta_2)(1 - \gamma_1 \gamma_2)$$

The discriminant of this biquadratic is

$$\Delta = (\delta_1 \gamma_1 - \delta_2 \gamma_2)^2 + 4(\delta_1 - \gamma_2)(\delta_2 - \gamma_1).$$

In order. Produce the diagram for all the cases (loxodromic, center-center, center-saddle, saddle-saddle) in parameter space $p'_1, q'_1, p'_2, q'_2, \theta$. 
Surfaces with antipodal symmetry
\((p_1 = p_2 = p, \ q_1 = q_2 = q)\)

\[ H(x, y) = h + \frac{1}{2} px^2 + \frac{1}{2} q y^2 \]

\[ A = \begin{bmatrix}
0 & \gamma & 0 & -1 \\
\delta & 0 & 1 & 0 \\
0 & -1 & 0 & \gamma \\
1 & 0 & \delta & 0 \\
\end{bmatrix} \]

where
\[ \gamma = -1 + 2q', \ \delta = 1 - 2p', \ p' = p/h, \ q' = q/h. \]
$A$ has two invariant subspaces of dimension 2:

- $V$ spanned by $v_1 = (1, 0, -1, 0), v_2 = (0, 1, 0, -1)$.
  
  \[ Av_1 = (\delta - 1)v_2 = -2p'v_2, \quad Av_2 = (1 + \gamma)v_1 = 2q'v_1 \]

- $W$ spanned by $w_1 = (1, 0, 1, 0), w_2 = (0, 1, 0, 1)$.
  
  \[ Aw_1 = (\delta + 1)w_2, \quad Aw_2 = (\gamma - 1)w_1 \]

The first subspace $V$ is tangent to the center manifold ($s_1 = -s_2$) while $W$ is transverse to it.
Surfaces with antipodal symmetry
\((p_1 = p_2 = p, \, q_1 = q_2 = q)\)

Proposition 1. On the invariant two dimensional submanifold \(S_{\text{ant}}\): if \(p \cdot q > 0\) we have a center. If \(p, \, q\) have opposite signs we have a saddle.

For the transverse subspace: if \((1 - p')(1 - q') > 0\) one has linear stability; if \((1 - p')(1 - q') < 0\) we have a saddle.

Proof. On the subspace \(V = \text{span}\{v_1, v_2\}\), the eigenvalues satisfy
\[
\lambda^2 = (\delta - 1)(1 + \gamma) = -4 \frac{pq}{h^2} = -4 p'q'.
\]
and on the subspace \(W = \text{span}\{w_1, w_2\}\), the eigenvalues satisfy
\[
\lambda^2 = (\gamma - 1)(\delta + 1) = -4 (1 - p')(1 - q'). \quad \Box
\]
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Questions on part II?

¡No contaban con mi astucia!
Surfaces of Revolution:
poles always stable
Surfaces of revolution \( (p_1 = q_1, p_2 = q_2) \)

\[
H_1 = h_1[1 + \frac{p_1'}{2}(x_1^2 + y_1^2)], \quad H_2 = h_2[1 + \frac{p_1'}{2}(x_2^2 + y_2^2)].
\]

Let the local profile of meridian a pole be

\[
z = \frac{x^2}{2\alpha} + \cdots
\]

Then for this pole we will show:

\[
p' = \frac{1}{2} - \frac{h^2}{2\alpha^2}, \quad h = \text{choose as needed}
\]
However:

$h_1$ and $h_2$ are related.

This is a global issue (more later).

To get them, we need to compute two integrals along the meridian.

(in a few slides)

Then adjust to make them equal.
Spectrum analysis for surfaces of revolution

Assume that a global conformal map was found, the quadratic expansions at the poles done, together with the \( \beta \)-adjustment. Substitute \( \gamma_1 = -\delta_1, \gamma_2 = -\delta_2 \) in \( p = \lambda^4 - 2\rho\lambda^2 + \kappa \).

\[ \rho_{\text{rev}} = -(1 + \frac{\gamma_1^2 + \gamma_2^2}{2}) < 0, \quad \kappa_{\text{rev}} = (1 - \gamma_1\gamma_2)^2 > 0. \]

A short calculation gives the discriminant

\[ \Delta_{\text{rev}} = \frac{(\gamma_1^2 - \gamma_2^2)^2}{4} + (\gamma_1 + \gamma_2)^2 > 0. \]

so the loxodromic case is ruled out. Thus we could have either a center-center or a center-saddle. Let us show that the center-saddle case is also ruled out. The eigenvalues satisfy

\[ \lambda^2 = \rho_{\text{rev}} \pm (\Delta_{\text{rev}})^{1/2} \]

Since \( \rho_{\text{rev}} < 0 \), at least one pair of eigenvalues is purely imaginary. In order to show that the other pair is also purely imaginary, we have to show that \( \rho^2 \geq \Delta_{\text{rev}} \). Indeed a short calculation gives

\[ \rho^2 - \Delta_{\text{rev}} = (1 - \gamma_1\gamma_2)^2. \]

Notice the indefinite case \( \gamma_1\gamma_2 = 1 \).
Proposition 2. For surfaces of revolution the pair of poles is always linearly stable (center-center). The complex eigenvalues are

$$\pm i \omega_1, \pm i \omega_2 \text{ with } \omega_1 \geq \omega_2 > 0$$

$$\omega_1 = \sqrt{\left(1 + \frac{\gamma_1^2 + \gamma_2^2}{2}\right) + \left(\frac{(\gamma_1^2 - \gamma_2^2)^2}{4} + (\gamma_1 + \gamma_2)^2\right)^{1/2}}$$

$$\omega_2 = \sqrt{\left(1 + \frac{\gamma_1^2 + \gamma_2^2}{2}\right) - \left(\frac{(\gamma_1^2 - \gamma_2^2)^2}{4} + (\gamma_1 + \gamma_2)^2\right)^{1/2}}$$
**Simple example: circular vortex billiard**

Inflate the double face unit disk to the sphere $S^2$ via the stereographic projection from the north pole in the down face, and from the south pole in the up face. The edges meet at the equator of the sphere.

The conformal factor from the planar (euclidian) metric to the sphere (round) metric at a point $(x, y, z) \in S^2$ is

$$h(x, y, z) = \frac{1}{1 + |z|}$$

For $z = \pm 1$:

$$h(x, y) = \frac{1}{2} + \frac{1}{8}(x^2 + y^2) + \cdots$$

$$h = \frac{1}{2}, \ p = q = \frac{1}{4} \Rightarrow \ p' = q' = \frac{1}{2}.$$
The equilibrium of the pair located at $x = y = 0, z = \pm 1$ is of center-center, with eigenvalues $\pm i$ both along the tangent space of the center manifold, and in the transverse subspace.

The frequencies are in 1:1 resonance, so we expect to observe interesting non linear behavior.
Surfaces of revolution, more

Visualize the surface sitting vertically over the x-y plane $P$, with the south pole at the origin. Let $s \in [0, L]$ the arc length along the meridian corresponding to $y = 0$, given by the function $x = x(s)$, such that $x(0) = x(L) = 0$, $x(s) > 0$ for $s \in (0, L)$. $z(s)$ can be recovered from $(dx/ds)^2 + (dz/ds)^2 = 1$.

Conformality of the surface $\Sigma$ to a plane $P : (r, \theta)$ means

$$ds^2 + x(s)^2 d\theta^2 = h^2(x) (dr^2 + r^2 d\theta^2)$$

Therefore, $ds/dr = x/r = h$ and one gets a separable ODE, in which the scale in the plane is arbitrary:

$$\frac{ds}{x(s)} = \frac{dr}{r}$$

Forcing a parallel $s_o$ to map into a circle of radius $r_o$ we have

$$r(s) = r_o \exp \left( \int_{s_o}^{s} \frac{ds}{x(s)} \right), \ s \in [0, L].$$
Profile near a pole:
\[ z = \frac{1}{2\alpha} x^2 + \cdots \quad \Rightarrow \quad p' = q' = p/h = \frac{1}{2} \left( 1 - \frac{h^2}{\alpha^2} \right). \]

Proof. We have \( s(x) = \int_0^x \sqrt{1 + (dz/dx)^2} \, dx \sim x + \frac{1}{6\alpha^2} x^3 + \cdots \) for small \( x \).

Example: ellipsoids of revolution \( \mathbb{E}(1, 1, c) : x^2 + y^2 + z^2/c = 1 \). The profile of the meridian of for small \( |x| \) is \( z = c^{1/2} \sqrt{1 - x^2} \sim c^{1/2} (1 - x^2/2) \) hence \( \alpha = 1/\sqrt{c} \).

For sphere, \( c = 1 \) so \( \alpha = 1 \). An oblate ellipsoid of revolution \( (c < 1) \) gives \( \alpha > 1 \).

Prolate ellipsoids of revolution to \( c > 1 \) so \( \alpha < 1 \).

\[ \text{Ansatz : } r(x) = (1/\rho) x \left( 1 + x^2/4\alpha^2 + O(x^4) \right) \]
\[ \rho = \text{scaling (undetermined).} \]

\[ \frac{x}{r} = \frac{h\Sigma/P}{\rho} = \frac{\rho}{1 + x^2/4\alpha^2} = \rho (1 - x^2/4\alpha^2). \]
\[ \frac{ds}{dr} = \frac{ds/dx}{dr/dx} = \rho \frac{1 + x^2/2\alpha^2}{1 + 3x^2/4\alpha^2} = \rho (1 - x^2/4\alpha^2) \]
The sphere $S^2$ representing the dynamics will live in the $(\tilde{x}, \tilde{y}, \tilde{z})$ space, while the original surface is in the $(x, y, z)$ space. We take the meridians in the corresponding $y = 0$, $\tilde{y} = 0$ planes. We have

$$h_{\Sigma/S^2} = h_{\Sigma/P} \cdot h_{P/S^2} = \rho \left(1 - x^2 / 4\alpha^2 \right) \left(\frac{1}{2} + \frac{1}{8} \tilde{x}^2\right).$$

We need to get the first order term of $x$ in terms of $\tilde{x}$. This can be easily achieved through the mediation of the radial coordinate $r$ in the plane. For the sphere,

$$r(\tilde{x}) = \frac{\tilde{x}}{1 - \tilde{z}} \sim \frac{1}{2} \tilde{x} + \cdots \text{ (south pole)}$$

and for the surface

$$x = \rho r + \cdots = \frac{\rho}{2} \tilde{x} + \cdots$$

Therefore,

$$h_{\Sigma/S^2} = \frac{\rho}{2} \left[1 + \left(\frac{1}{4} - \frac{\rho^2}{16\alpha^2}\right) \left(\tilde{x}^2 + \tilde{y}^2\right) + \cdots\right], \quad h = \rho/2$$

Done!
Proposition 3. Let the meridian profile of a surface of revolution near a pole be \( z = (1/2\alpha)x^2 + \cdots \). Consider the conformal map from the surface to the unit sphere such that the conformal factor at the pole is \( h \). Then

\[
p' = q' = p/h = \frac{1}{2} \left( 1 - \frac{h^2}{\alpha^2} \right).
\]

Example. The profile for a sphere \( \Sigma_a \) of radius \( a \) is

\[
z = \sqrt{a^2 - x^2} \sim a - x^2/2a
\]

so \( \alpha = a \), and we get \( p'_1 = q'_1 = \frac{1}{2} \left( 1 - \frac{h^2}{\alpha^2} \right) \).

Check: \( \Sigma = S^2 \) in the \((x, y, z)\) space, the conformal map to the sphere \( S^2 \) in the \((\tilde{x}, \tilde{y}, \tilde{z})\) space that sends equator to equator is the identity.

Then \( \rho = 2, h = 1, \alpha = 1 \), so we get \( q' = p' = 0 \).
Spheroids (no computations needed!)

Since there is symmetry with respect to the equatorial plane, we can assume $h_1 = h_2 = h$. Moreover, the local expansions at the two poles coinciding \textit{apriori}. The value of $h$ is unknown. At the two poles of the spheroid, at least we can say that

$$p' = p/h = \frac{1}{2} (1 - \frac{h^2}{\alpha^2}) \leq \frac{1}{2}.$$ 

The fact that $p' \leq \frac{1}{2}$ is bounded in a “physical” surface of revolution in $\mathbb{R}^3$ is not unexpected.

For surfaces in $\mathbb{R}^3$ the arc length along a meridian, starting at a pole, must satisfy $s(x) > x$, for $x \neq 0$. But an abstractly produced $S^1$ equivariant metric can violate this condition, take for instance $\alpha d\phi^2 + \cos^2 \phi d\theta^2$, with $0 < \alpha < 1$.

This illustrates the depth of Gromov $C^\infty$ (or analytic) embedding results. For instance, any compact Riemann surface with a $C^\infty$ (or analytic) metric $g$ can be $C^\infty$ (or analytically) isometrically embedded in $\mathbb{R}^5$.
Surfaces of revolution: $h$ at the poles

Meridian: $x = x(t), z = z(t), t \in [a, b]$

$x(t) > 0 \in (a, b), x(a) = x(b) = 0, \dot{x}(a) > 0, \dot{x}(b) < 0, z(a) \neq z(b), \dot{z}(a) = \dot{z}(b) = 0$.

$$r(t) = r_o \exp \left( \int_{t_o}^{t} \frac{\sqrt{(\dot{x}(t))^2 + (\dot{z}(t))^2}}{x(t)} \, dt \right)$$

Increasing function, with $r(a) = 0, r(b) = \infty$. We split

$$\frac{\sqrt{\dot{x}^2 + \dot{z}^2}}{x} = \frac{\dot{x}}{x} + \left( \frac{\sqrt{\dot{x}^2 + \dot{z}^2} - \dot{x}}{x} \right).$$

We denote the second term by $\alpha(t)$.

$$\alpha(t) = \frac{\sqrt{\dot{x}^2 + \dot{z}^2} - \dot{x}}{x} = \frac{\dot{z}^2}{\sqrt{\dot{x}^2 + \dot{z}^2} + \dot{x}} \geq 0$$
Assume that $z^2/x$ is finite at the poles. To make the expressions simpler, let us assume that $x(t)$ is increasing for $t \in (a, c)$ and decreasing for $t \in (c, b)$ for some $c$ in the interval.

Take $t_o = c, r_o = 1$ so that the parallel corresponding to $t = c$ will be sent to the unit circle.

$$r(t) = \begin{cases} 
\frac{A(t) x(t)}{x(c)}, & a \leq t \leq c \\
\frac{B(t) x(c)}{x(t)}, & c \leq t \leq b
\end{cases}$$

$$A(t) = \exp \left( \int_a^c \alpha(t) \, dt \right), \quad B(t) = \exp \left( \int_c^t \alpha(t) \, dt \right)$$

Near the poles,

$$r \sim \frac{A}{x(c)} \cdot x \text{ (south)}, \quad r \sim \frac{B x(c)}{x} \text{ (north)}.$$

where

$$A = A(a) \exp \left( \int_a^c \alpha(t) \, dt \right), \quad B = B(b) = \exp \left( \int_c^b \alpha(t) \, dt \right)$$
**Proposition 4.** *The conformal factors between the surface to the sphere at the poles are:*

\[ h_{\Sigma - S^2}(\text{south}) = \frac{x(c)}{2A}, \quad h_{\Sigma - S^2}(\text{north}) = \frac{x(c)B}{2} \]

so the adjusting factor is

\[ \beta^2 = AB. \]

**Proof.** The conformal factor from the surface of revolution to the plane is \( h_{\Sigma / P} = x(t)/r(t) \). Therefore

\[ h_{\Sigma / P}(\text{south}) = \frac{x(c)}{A}, \quad h_{\Sigma / P}(\text{north}) \sim \frac{x(c)B}{r^2}. \]

Finally, compose with the inverse of the stereographic projection of the unit sphere, from the north pole to the equatorial plane. Let the unit sphere be in \((x_1, x_2, x_3)\) space. Along \(x_2 \equiv 0\) it is given by \(x_1 = 2r/(1 + r^2)\) so the conformal factor from the plane to the sphere is \( h_{P / S^2} = \frac{r}{x_1} = \frac{1+r^2}{2} \). Just multiply and we are done. \(\Box\)
Part III:
Some computable examples
Surfaces of revolution: computable examples

- Ellipsoid of revolution: elementary integrals

- Mr. Bean surface*: can do with elliptic integrals

\[ z = a \sin^2 t + b \cos t, \quad x = \sin t, \quad t \in [0, \pi]. \]

\( a = 0 \) is the ellipsoid of revolution with semi-axis 1, 1, b)

Summarizing: The local expansions are easy to obtain. The only real work is to compute \( A, B \) for the adjusting \( \beta = \sqrt{AB} \). The frequencies \( \omega_1, \omega_2 \) can be obtained analytically from the formulas in the previous slide.

*Dritschel/Boatto, RSPA, 2015, DOI: 10.1098/rspa.2014.0890
Matrioschka (homework)

All the calculations can be done with very elementary integrals.

Three parameters: the two radii, \( a > b \) and \( 0 < \theta < \pi/2 \), angle between the generating line and the axis of symmetry.
The doubled planar elliptical region

In 1869 H.A. Schwarz gave the conformal map from the unit disk $D$ in the complex plane $\omega$ to the interior of an ellipse in the $\xi$-plane:

$$\omega \mapsto \xi = \sin \left[ \frac{\pi}{2K(r)} F\left( \frac{\omega}{\sqrt{r}} ; r \right) \right]$$

$F$ is the incomplete elliptic integral of the first kind

$$F(w; r) = \int_{0}^{\omega} \frac{dt}{\sqrt{(1 - t^2)(1 - r^2 t^2)}}.$$

$$\frac{K\left(\sqrt{1 - r^2}\right)}{K(r)} = \frac{4\tau}{\pi}, \quad r \in (0, 1), \quad \tau \in (0, \infty)$$

involving the complete (real) integral $K(r) = F(1, r)$.

Ellipse semiaxis: $c = \cosh \tau > b = \sinh \tau$. Foci at $\pm 1$ in the $\xi$-plane.

As $r \to 1$ then $\tau \to 0$ (the segment).

As $r \to 0$, then $\tau \to \infty$ (a very large disk).

We can use either $r \in (0, 1)$ or $\tau \in (0, \infty)$ to parametrize the ellipse.
The mapping \( w = \text{sn}(u, k) \)
The south pole of the unit sphere of the \((x_1, x_2, x_3)\) space is sent to the origin of \(\omega\)-plane by stereographic projection from \((0, 0, 1)\).

\(h(x_1, x_2, x_3)\) is the conformal factor of the composition
\[
s = (x_1, x_2, x_3) \xrightarrow{\text{stereog}} \omega \xrightarrow{\text{Schwarz}} (\text{disk}) \xrightarrow{} (\text{ellipse}).
\]

Expand the integrand of the incomplete \(F(\cdot, r)\)
\[
\frac{1}{\sqrt{(1-t^2)(1-r^2t^2)}} \sim 1 + \frac{1}{2} (1+r^2)t^2 \Rightarrow F\left(\frac{w}{\sqrt{r}}; r\right) \sim \frac{w}{\sqrt{r}} + \frac{1}{6} (1+r^2) \frac{w^3}{r^{3/2}}
\]

Denote for short, \(K = K(r)\). Inserting in the sine series:
\[
\xi = \frac{\pi}{2K} \left[ \frac{w}{\sqrt{r}} + \frac{1}{6} (1+r^2) \frac{w^3}{r^{3/2}} \right] - \frac{1}{6} \left( \frac{\pi}{2K} \right)^3 \frac{w^3}{r^{3/2}}
\]

Collecting the cubic terms, we get ....
\[ \xi = \frac{\pi}{2K(r)\sqrt{r}} \left( w + \frac{1}{3} M w^3 \right) + \cdots \]

where

\[ M = M(r) = \frac{1}{2r} \left( 1 + r^2 - \frac{\pi^2}{4K^2(r)} \right). \]

Then

\[ h_{R_r/D}^2 = \frac{d\xi}{dw} \cdot \frac{d\xi}{dw} = \left( \frac{\pi/2}{K\sqrt{r}} \right)^2 \left[ 1 + M(w^2 + \overline{w}^2) \right] = \left( \frac{\pi/2}{K\sqrt{r}} \right)^2 \left[ 1 + 2M(X_3^2 - X_2^2) \right] \]

Therefore

\[ h_{R_r/S^2} = \frac{\pi}{2K\sqrt{r}} \left[ 1 + M(X_3^2 - X_2^2) \right] \cdot \frac{1}{2} \left( 1 + \frac{1}{4}(x_2^2 + x_3^2) \right) \]

Near \((1, 0, 0)\), we have \(X_3 \sim x_3/2\), \(X_2 \sim x_2/2\) so that

\[ h_{R_r/S^2} = \frac{\pi}{2K\sqrt{r}} \left[ 1 + \frac{M}{4}(x_3^2 - x_2^2) \right] \cdot \frac{1}{2} \left( 1 + \frac{1}{4}(x_2^2 + x_3^2) \right) \]

Collecting terms...
The doubled planar elliptical region: conclusion

\[ h_{R_z/S^2} = \frac{\pi}{4K\sqrt{r}} \left[ 1 + \frac{p'}{2}x_3^2 + \frac{q'}{2}x_2^2 \right]. \]

\[ p' = \frac{1+M}{2}, \quad q' = \frac{1-M}{2}, \quad M = \frac{1}{2r}\left(1 + r^2 - \frac{\pi^2/4}{K^2(r)}\right). \]

\[ p'q' = (1-p')(1-q') = \frac{1-M^2}{4}. \]

Proposition 1 implies: if \(|M| > 1\) we will have center-center, and if \(|M| < 1\), saddle-saddle.

In the limit \(r \to 0\) (circle): since \(K(0) = \pi/2\) we get \(M \to 0\), so \(p' = q' = 1/2\)

In the limit \(r \to 1\): we get \(M \to 1\) so \(p' = 1, q' = 0\).

The table of values of \(M(r)\) shows that the antipodal pair \((0^u, 0^b)\) of the ellipse is always center-center, for all \(r \in [0, 1]\).
<table>
<thead>
<tr>
<th>r</th>
<th>K(r)</th>
<th>M(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.574745562</td>
<td>0.07504711</td>
</tr>
<tr>
<td>0.2</td>
<td>1.586867847</td>
<td>0.1503827</td>
</tr>
<tr>
<td>0.3</td>
<td>1.60804862</td>
<td>0.22632604</td>
</tr>
<tr>
<td>0.4</td>
<td>1.639999866</td>
<td>0.30326745</td>
</tr>
<tr>
<td>0.5</td>
<td>1.685750355</td>
<td>0.38173312</td>
</tr>
<tr>
<td>0.6</td>
<td>1.750753803</td>
<td>0.46250971</td>
</tr>
<tr>
<td>0.7</td>
<td>1.845693998</td>
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<tr>
<td>0.8</td>
<td>1.995302778</td>
<td>0.63765125</td>
</tr>
<tr>
<td>0.9</td>
<td>2.280549138</td>
<td>0.74199017</td>
</tr>
<tr>
<td>0.95</td>
<td>2.590011231</td>
<td>0.80772582</td>
</tr>
<tr>
<td>0.99</td>
<td>3.356600523</td>
<td>0.88944538</td>
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<tr>
<td>0.995</td>
<td>3.696875082</td>
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</tr>
<tr>
<td>0.999</td>
<td>4.495596396</td>
<td>0.9388965</td>
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<tr>
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<tr>
<td>0.9999</td>
<td>5.645148217</td>
<td>0.96128293</td>
</tr>
<tr>
<td>1</td>
<td>11.40135369</td>
<td>0.99050933</td>
</tr>
</tbody>
</table>

K(r) evaluated from https://keisan.casio.com/exec/system/1180573451

Values of M(r) where computed on an Excel Workbook.


## Triaxial ellipsoid

\[
\begin{align*}
    h_A &= \sqrt{\frac{c-b}{I_3-I_2}}, \\
    p'_A &= \frac{p_A}{h_A} = \frac{1}{I_3-I_2} \left[ \frac{b-a}{b} - (I_2-I_1) \right], \\
    q'_A &= \frac{q_A}{h_A} = \frac{1}{I_3-I_2} \left[ (I_3-I_1) - \frac{c-a}{c} \right].
\end{align*}
\]

Formulas at vertices \( B \) and \( C \) are analogous.

Recall: \( I_1, I_2, I_3 \) are functions of \( a, b, c \) via “Master equation”.
<p>| | | | | | | |</p>
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<tr>
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<tbody>
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<td>a</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
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<td>6</td>
<td>1.1</td>
<td>2</td>
<td>4</td>
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</tr>
<tr>
<td>c</td>
<td>1.02</td>
<td>9</td>
<td>9</td>
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<tbody>
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<td>l1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>l2</td>
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<td>1.4052</td>
<td>1.001</td>
<td>1.0229</td>
<td>1.1628</td>
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<tr>
<td>l3</td>
<td>1.0198</td>
<td>2.6325</td>
<td>8.5734</td>
<td>6.0466</td>
<td>3.6638</td>
<td></td>
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</tr>
</thead>
<tbody>
<tr>
<td>p'A</td>
<td>0.0098</td>
<td>0.3488</td>
<td>0.0119</td>
<td>0.095</td>
<td>0.2348</td>
<td></td>
</tr>
<tr>
<td>q'A</td>
<td>0.0195</td>
<td>0.6059</td>
<td>0.8827</td>
<td>0.8276</td>
<td>0.7097</td>
<td></td>
</tr>
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</thead>
<tbody>
<tr>
<td>p'B</td>
<td>0.0099</td>
<td>0.5476</td>
<td>0.884</td>
<td>0.8413</td>
<td>0.7303</td>
<td></td>
</tr>
<tr>
<td>q'B</td>
<td>-0.0099</td>
<td>-2.8146</td>
<td>-0.0131</td>
<td>-0.1936</td>
<td>-1.0651</td>
<td></td>
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</thead>
<tbody>
<tr>
<td>p'C</td>
<td>-0.0201</td>
<td>-15.7129</td>
<td>-436.0694</td>
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<td>q'C</td>
<td>-0.01</td>
<td>-1.7946</td>
<td>-399.2462</td>
<td>-66.4068</td>
<td>-7.6864</td>
<td></td>
</tr>
</tbody>
</table>

Values used in A. Regis thesis, $I_1 = a$ for convenience.

In preparation: a thorough sampling of the parameter region

$$0 < a < b < c = 1.$$
Final remarks

• Embedding in $\mathbb{R}^3$ (Gromov: can do in $\mathbb{R}^5$)

• Surfaces of revolution: rings of vortices

• Triaxial ellipsoid: umbilical points?

• Surfaces with antipodal symmetry: transverse behavior to $S_{\text{ant}}$ - Floquet multipliers.
Congrats James!!
Additional material
Conformality at the umbilical points

Umbilics correspond to $\lambda_1 = \lambda_2 = b$, and at the sphere we have $\mu_1 = \mu_2 = I_2$. Therefore we get a $0/0$ indeterminacy.

Let $\alpha, \beta > 0$, with $\alpha + \beta = 1$, and take

$$\mu_1 = I_2 - \alpha \epsilon + O(\epsilon^2), \quad \mu_2 = I_2 + \beta \epsilon + O(\epsilon^2), \quad \epsilon > 0.$$ 

Let's examine the limit

$$\lim_{\epsilon \to 0^+} \frac{(Q^{-1}T(I_2 + \beta \epsilon) - P^{-1}S(I_2 - \alpha \epsilon))}{\epsilon}$$

Using l'Hôpital, we should investigate

$$\alpha \left[ \lim_{\mu_1 \to I_2} \frac{d}{d\mu_1} P^{-1}S(\mu_1) \right] + \beta \left[ \lim_{\mu_2 \to I_2} \frac{d}{d\mu_2} Q^{-1}T(\mu_2) \right]$$

We do it for the first:

$$L = \lim_{\mu_1 \to I_2} \frac{d}{d\mu_1}[P^{-1}S(\mu_1)]$$
We have:

\[
\frac{d}{d\mu_1} P^{-1} S(\mu_1) = \frac{dP^{-1}}{du} \bigg|_{u=S(\mu_1)} \cdot \frac{dS}{d\mu_1} = \frac{1}{\frac{dP}{d\lambda_1}} \cdot \frac{dS}{d\mu_1}
\]

where \(dP/d\lambda_1\) is computed at \(\lambda_1 = P^{-1} S(\mu_1)\). Therefore

\[
\frac{d}{d\mu_1} P^{-1} S(\mu_1) = \left[ \frac{1/((I_2 - \mu_1)(I_2 - \mu_1)(I_3 - \mu_1))}{P^{-1} S(\mu_1)/(P^{-1} S(\mu_1) - a)(b - P^{-1} S(\mu_1))(c - P^{-1} S(\mu_1))} \right]^{1/2}
\]

Pulling out (if we may) the factors that have a direct limit we get

\[
L = \left[ \frac{(b - a)(c - b)}{b(I_2 - I_1)(I_3 - I_2)} \right]^{1/2} \lim_{\mu_1 \to I_2} \frac{1/\sqrt{I_2 - \mu_1}}{1/\sqrt{b - P^{-1} S(\mu_1)}}
\]

The limit in the right is, by a stroke of luck, \(\sqrt{L}\). Hence

\[
L = \left[ \frac{(b - a)(c - b)}{b(I_2 - I_1)(I_3 - I_2)} \right]^{1/2}
\]

The second limit is computed analogously and gives, seemingly by another stroke of luck, the same result, but this is indeed what we expect to happen by the theory of Riemann surfaces.
Vortices on the sphere
history and recent developments

Gromeka (1851-1889, MR0056525), Zermelo (1899)
Bogomolov (1977), Kimura/Okamoto (1987),

Great number of papers in modern times:

Aref, Borisov/Mamaev, Cabral, Dritschel, Kidambi,
Marsden, Montaldi, Newton, Patrick, Pekarsky, Polvani,
Roberts, Schmidt, Simó, Tronin, ...

(sorry for MANY omissions)
Vortices on compact surfaces $\Sigma$ (any genus)
default (but not mandatory): constant curvature $(G_o, R_o)$.

\[
H = \sum_{1 \leq i < j \leq N} \kappa_i \kappa_j G_g(s_i, s_j) + \sum_{\ell=1}^N \frac{1}{2} \kappa_\ell^2 R_g(s_\ell)
\]

\[
\Omega_{\text{collective}}(s_1, \ldots, s_N) = \sum_{\ell=1}^N \kappa_\ell \Omega(s_\ell)
\]

$G_g(\sigma_1, \sigma_2)$ = Green function of Laplace-Beltrami operator

\[
R_g(\sigma) = \lim_{\sigma' \to \sigma} G(\sigma', \sigma) - \frac{1}{2\pi} \ln d(\sigma', \sigma)
\]

(Robin function)

$\Omega_g$ = area form of the metric $g$.

Boatto/Koiller, Vortices on Closed Surfaces, Fields Institute 73, 2015
https://link.springer.com/chapter/10.1007/978-1-4939-2441-7_10
Green function

\[
\Delta_g G(\sigma, \sigma_o) = -\frac{1}{\text{Area}(\Sigma)} + \delta(\sigma, \sigma_o)
\]

\[
G(\sigma, \sigma_o) - \log d(\sigma, \sigma_o)/2\pi \text{ bounded}
\]

\[
\int_\Sigma G(\sigma, \sigma_o) \Omega(\sigma) = 0, \ G(\sigma, \sigma_o) = G(\sigma_o, \sigma).
\]

There are transformation formulas between Green and Robin functions of metrics \( g, \tilde{g} \) in the same conformal class, involving \( \Delta_g^{-1} \) (or \( \tilde{g} \)).

Metrics in the same conformal class: \( g = h^2 g_0 \)

The dynamics will be transported to the constant curvature model

\((\Sigma_o, g_o)\) with its data \((G_o, R_o, \Omega_o)\).

- In the new collective symplectic form use \( h^2(s) \Omega_o \), the pull back of the area form of \((\Sigma, g)\).

- Add to the Hamiltonian \( H_o(s_1, ..., s_N) \) for vortices in the metric with constant curvature (that uses \( G_o \) and \( R_o \)) two new terms:

\[
-\frac{1}{4\pi} \sum_j \kappa_j \ln h(s_j)
\]

\[
\left( \frac{\sum \kappa_j}{\text{Area}_g} \right) \sum_j \kappa_j (\Delta_{g^{-1}} h^2)(s_j)
\]

(The second disappears when \( \sum_j \kappa_j = 0 \).)
Vortex pairs: $\Sigma \times \Sigma$, $\kappa_1 = -\kappa_2 = 1$

Symplectic form: $\Omega(\sigma_1) - \Omega(\sigma_2)$

Hamiltonian

$$H = -G(\sigma_1, \sigma_2) + \frac{1}{2} (R(\sigma_1) + R(\sigma_2))$$

Can be rewritten as follows:

$$H = -\frac{1}{2\pi} \ln d(\sigma_1, \sigma_2) + B$$

where $B = O((d(\sigma_1, \sigma_2))^2)$ is the Batman function

$$B = \frac{1}{2} (R(\sigma_1) + R(\sigma_2)) - \left[ G(\sigma_1, \sigma_2) - \frac{1}{2\pi} \ln d(\sigma_1, \sigma_2) \right]$$
Kimura's conjecture: “Vortex pairs do geodesics”

Center arrow map \( (p, \sigma) \in T^*\Sigma \rightarrow (\sigma_1, \sigma_2) \in \Sigma \times \Sigma \)

\[
p \xleftarrow{\text{Leg}} v \ , \ \sigma_{1,2} = \exp(\sigma, \pm J(\epsilon \frac{v}{2}))
\]

One can show:

\[
\Omega(\sigma_1) - \Omega(\sigma_2) = \epsilon^2 \Omega^\text{can}_{T^*\Sigma} + O(\epsilon^4)
\]

Dominant term in Hamiltonian: \(-\ln d(\sigma_1, \sigma_2) = -\ln |v|\).
In order: compute the deformation term

\[
\Omega(\sigma_1) - \Omega(\sigma_2) = \epsilon^2 \Omega_{T^*\Sigma}^{can} + O(\epsilon^4)
\]

\[ p_{\sigma} \in T^*\Sigma \rightarrow v_{\sigma} \in T\Sigma \rightarrow w_{\sigma} = J(v_{\sigma}/2) \rightarrow (\sigma_1, \sigma_2) = (\exp(-\epsilon w_{\sigma}), \exp(\epsilon w_{\sigma})) \]
Green function of Laplace-Beltrami operator
for surfaces of constant curvature

for sphere: $G = \frac{\text{log of the euclidian distance}}{2\pi}$

genus 1 (all tori): $G$ is known, via elliptic functions

genus $\geq 2$ still in infancy (H. Avelin’s thesis)

For sphere ($K \equiv 1$) and genus 1 surfaces ($K \equiv 0$)

$R$ is constant

https://projecteuclid.org/euclid.em/1317758094
But not for genus $\geq 2$

Level sets of Robin function on Bolza’s surface with $K = -1$
(Fundamental domain)

http://dx.doi.org/10.1098/rspa.2017.0447
A canonical metric for Riemann surfaces: “steady vortex metric”

A single vortex does not move in the Euclidean plane, in the round sphere, and in any flat torus. A metric for which a single vortex does not move may be called a steady vortex metric.

For a non-compact Riemann surface with N ends there is a Riemannian metric g compatible with the conformal structure of such that its associated Robin function is constant. It is unique up to a multiplicative constant.


https://link.springer.com/article/10.1007/s00332-017-9380-7
Master equation: the relation between \((a, b, c)\) and \((I_1, I_2, I_3)\)

Both \(k_1, k_2\) and \(\ell_1, \ell_2\) are complementary: \(k_1^2 + k_2^2 = 1, \ell_1^2 + \ell_2^2 = 1\).

We denote

\[
K(k) = F(\frac{\pi}{2}, k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad 0 \leq k \leq 1
\]

the complete integral of the first type.

We now fulfill the requirement that the umbilical points of the ellipsoid should correspond to the special points in the sphere so that the map between \(\mathbb{E}^2(a, b, c)\) and \(S^2\) is well defined. This amounts the two equalities

\[
K(k_1) = \frac{\sqrt{I_3 - I_1}}{2} P(b), \quad K(k_2) = \frac{\sqrt{I_3 - I_1}}{2} Q(b)
\]

where we have the complete integrals of the third kind

\[
P(b) = \frac{2a}{\sqrt{b(c-a)}} \Pi(\pi/2, \ell_1, n_1)
\]

\[
Q(b) = \frac{2c}{\sqrt{b(c-a)}} \Pi(\pi/2, \ell_2, n_2).
\]
Proposition 5. *(Master Equation)*

\[
\frac{K \left( \sqrt{1 - k_1^2} \right)}{K(k_1)} = \frac{Q(b)}{P(b)} = \frac{c \ \pi(\pi/2, \ell_2, n_2)}{a \ \pi(\pi/2, \ell_1, n_1)}
\]

After getting \( k_1 \), the parameters \( I_1, I_2 \) and \( I_3 \) are obtained from

\[
I_3 - I_1 = 4 \left( \frac{K(k_1)}{P(b)} \right)^2 \quad \left[ = 4 \left( \frac{K(k_2)}{Q(b)} \right)^2 \right]
\]

\[
I_2 - I_1 = (I_3 - I_1) k_1^2, \quad (I_3 - I_2 = (I_3 - I_1) k_2^2 \text{ is redundant})
\]

No harm is done by setting \( I_1 = 0 \) for simplicity.

The left hand side of the Master Equation decreases from \( \infty \) to 0 as \( k_1 \) runs from 0 to 1. Thus there is an unique solution to this equation.

The special points in the sphere corresponding to the umbilics are

\(( \pm k_1, 0, \pm k_2 )\).
About the equation $K \left( \sqrt{1 - k^2} \right) / K(k) = \sqrt{r}$

The solution $k = \Lambda(r)$ of this equation is called the \textit{elliptic lambda function} and can be obtained via Jacobi theta functions

$$k = \left[ \frac{\theta_2(0, qr)}{\theta_3(0, qr)} \right]^2$$

$q_r = \exp(-\pi/\sqrt{r})$ is called the \textit{nome} and the theta functions are

$$\theta_2(0, q) = \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2}, \quad \theta_3(0, q) = \sum_{m=-\infty}^{\infty} q^{m^2}.$$
Jacobi’s remarkable coordinates, 1838

Jacobi’s used confocal quadrics coordinates on

\[ \mathcal{E}(a, b, c) : \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \]

to show that the geodesics are integrable.

This problem, as we know, was the first example of a completely integrable two degrees of freedom system without geometric symmetry.
“Ich habe vorgestern die geodätische Linie für ein Ellipsoid mit drei ungleichen Achsen auf Quadraturen zurückgeführt. Es sind die einfachsten Formeln von der Welt, Abelsche Integrale, die ich in die bekannten elliptischen verwandeln, wenn man 2 Achsen gleich setzt.”

The day before yesterday, I reduced to quadrature the problem of geodesic lines on an ellipsoid with three unequal axes. They are the simplest formulas in the world, Abelian integrals, which become the well known elliptic integrals if 2 axes are set equal.”
Jacobi, C. G. J., Note von der geodätischen Linie auf einem Ellipsoid und den verschiedenen Anwendungen einer merkwürdigen analytischen Substitution J. Crelle 19, 309–313 (1839)

(The geodesic on an ellipsoid and various applications of a remarkable analytical substitution)
15.

Note von der geodätischen Linie auf einem Ellipsoid und den verschiedenen Anwendungen einer merkwürdigen analytischen Substitution.

(Von Herrn C. G. J. Jacobi, Professor ordin. zu Königsberg in Pr.)

(gelesen in der Königl. Akademie der Wissenschaften zu Berlin am 18. April 1839.)


\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,
\]

und \(a\) die kleinste, \(b\) die mittlere, \(c\) die größte der drei Constanten \(a, b, c\). Da man die drei Coordinaten des Punctes einer gegebenen Oberfläche durch zwei Größen ausdrücken kann, so wähle ich hierzu die Winkel \(\Phi\) und \(\Psi\), welche die Coordinaten durch die Formeln bestimmen,

\[
x = \sqrt{\frac{a}{c-a}} \sin \Phi \sqrt{\sin^2 \Psi + c \sin^2 \Psi - a),
\]

\[
y = \sqrt{b \cos \Phi \sin \Psi},
\]

\[
z = \sqrt{\frac{c}{c-a}} \cos \Psi \sqrt{c - a \cos^2 \Phi - b \sin^2 \Phi}).
\]
• Jacobi also showed how to construct \textit{local} isothermal coordinates.

• I will show how to make these coordinates global: we get an explicitly computable conformal map from the ellipsoid $\mathcal{E}(a, b, c)$ to the sphere $S^2$.

(It is unique up to Moebius transformations of $S^2$.)

Schering, E., Über die conforme abbildung des ellipsoids auf der ebene,
Gesammelte Mathematische Werke, ch. III, Mayer and Muller, Berlin (1902)
(available in \url{http://name.umdl.umich.edu/AAT1702.0001.001})

Confocal quadrics coordinates

\[ E(a, b, c) : \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1. \]

Consider the equation with \((x, y, z) \in E(a, b, c)\):

\[ \frac{x^2}{a - \lambda} + \frac{y^2}{b - \lambda} + \frac{z^2}{c - \lambda} = 1. \]
Jacobi’s confocal coordinates: \( a \leq \lambda_1 \leq b \leq \lambda_2 \leq c \).

\[
x^2 = \frac{a(a - \lambda_1)(a - \lambda_2)}{(a - b)(a - c)},
\]

\[
y^2 = \frac{b(b - \lambda_1)(b - \lambda_2)}{(b - a)(b - c)},
\]

\[
z^2 = \frac{c(c - \lambda_1)(c - \lambda_2)}{(c - a)(c - b)}.
\]
Umbilical points

Each octant is parametrized by \((\lambda_1, \lambda_2) \in [a, b] \times [b, c]\).

\[
\lambda_1 = b, \lambda_2 = c \Rightarrow \left(\sqrt{a}, 0, 0\right)
\]

\[
\lambda_1 = a, \lambda_2 = c \Rightarrow \left(0, \sqrt{b}, 0\right)
\]

\[
\lambda_1 = a, \lambda_2 = b \Rightarrow \left(0, 0, \sqrt{c}\right)
\]

There are four umbilical points located in the middle ellipse \((y = 0)\):

\[
\left(\pm \sqrt{\frac{a(b - a)}{c - a}}, 0, \pm \sqrt{\frac{c(c - b)}{c - a}}\right)
\]

corresponding to \(\lambda_1 = \lambda_2 = b\).
Metric

Proposition 1. (Jacobi) The metric $ds^2$ induced by the embedding of the ellipsoid $E(a, b, c)$ in $\mathbb{R}^3$ is

$$ds^2 = \frac{\lambda_2 - \lambda_1}{4} \left[ \frac{\lambda_1 d\lambda_1^2}{(\lambda_1 - a)(\lambda_1 - b)(\lambda_1 - c)} + \frac{-\lambda_2 d\lambda_2^2}{(\lambda_2 - a)(\lambda_2 - b)(\lambda_2 - c)} \right]$$

Making the pair of reparametrizations (more soon!)

$$u = \int_a^{\lambda_1} \sqrt{\ldots} d\lambda_1, \quad v = \int_{\lambda_2}^{c} \sqrt{\ldots} d\lambda_2$$

the metric becomes of Liouville type

$$ds^2 = (f(u) + g(v))(du^2 + dv^2)$$
$(u, v)$ are isothermal coordinates with a singularity in $U$
Complete integrability

For a Liouville metric

\[ ds^2 = (f(u) + g(v))(du^2 + dv^2) \]

the geodesics satisfy

\[ \int \frac{du}{\sqrt{f(u) + \gamma}} \pm \int \frac{dv}{\sqrt{g(v) - \gamma}} = \delta \]

where \( \gamma, \delta \) are constants.

Two-dimensional Riemannian metrics with integrable geodesic flows. Local and
http://dx.doi.org/10.1070/SM1998v189n10ABEH000346
Reparametrizations

Elliptic integrals of the third kind

\[ \Pi(\phi, k, n) = \int_0^\phi \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}}. \]

Define \( u = P(\lambda_1) \) (increasing) and \( v = Q(\lambda_2) \) (decreasing)

\[ u = P(\lambda_1) = \int_a^{\lambda_1} \sqrt{\frac{t}{(t-a)(t-b)(t-c)}} \, dt = \frac{2a}{\sqrt{b(c-a)}} \Pi(\phi, k, n) \]

with

\[ \phi = \arcsin \sqrt{\frac{b(\lambda_1 - a)}{\lambda_1(b-a)}} , \quad k = \sqrt{\frac{c(b-a)}{b(c-a)}} \quad \text{and} \quad n = \frac{a-b}{b}; \]

\[ v = Q(\lambda_2) = \int_{\lambda_2}^{c} \sqrt{\frac{-t}{(t-a)(t-b)(t-c)}} \, dt = \frac{2c}{\sqrt{b(c-a)}} \Pi(\phi, k, n) \]

with

\[ \phi = \arcsin \sqrt{\frac{b(c-\lambda_2)}{\lambda_2(c-b)}} , \quad k = \sqrt{\frac{a(c-b)}{b(c-a)}} \quad \text{and} \quad n = \frac{c-b}{b}. \]
Proposition 2. The metric has, in the first octant, the isothermal coordinates \((u, v) \in [0, P(b)] \times [0, Q(b)]\),

\[
ds^2 = h^2(u, v)(du^2 + dv^2)
\]

with

\[
h^2(u, v) = \frac{\lambda_2(v) - \lambda_1(u)}{4}.
\]

The conformal map becomes singular (because the conformal factor vanishes) when \(\lambda_2(v) = \lambda_1(u)\).

This occurs only for \(u = P(b), v = Q(b)\), precisely the umbilical point.
Lines of curvature of the triaxial ellipsoid. Cuts along the top and bottom segments joining the umbilical points results (topologically) on an open cylinder. One could as well make the cuts sidewise.
Double branched cover of the torus over the ellipsoid.
<table>
<thead>
<tr>
<th>C2</th>
<th>U2</th>
<th>C</th>
<th>U1</th>
<th>C1</th>
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<tbody>
<tr>
<td>x&lt;0</td>
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</table>

- C2
- U2
- C
- U1
- C1
Our construction of a conformal map from the ellipsoid to the sphere makes use of confocal coordinates in \( \mathbb{E} \) and spherico-conical coordinates in \( \mathbb{S}^2 \), that has four “fake” singular points.

The umbilical points on the ellipsoid are mapped into the singular points of the spherico-conical coordinates. This coordinate system depends on three arbitrary parameters \( I_1 < I_2 < I_3 \), that will be chosen so that one octant of the ellipsoid gets mapped exactly into one octant of the sphere.

*This will permit to extend the conformal transformation between the whole surfaces, in such a way that the coordinate lines distributions correspond.*
Felix Klein, On Riemann’s Theory Of Algebraic Functions

All closed surfaces which can be conformally represented upon each other by means of a uniform correspondence, are, of course, to be regarded as equivalent for our purposes. For every complex function of position on the one surface will be changed by this representation into a similar function on the other surface; hence, the analytical relation which is graphically expressed by the co-existence of two complex functions on the one surface is entirely unaffected by the transition to the other surface. For instance, the ellipsoid may be conformally represented, by virtue of known investigations, on a sphere, in such a way that each point of the former corresponds to one and only one point of the latter; this shows us that the ellipsoid is as suitable for the representation of rational functions and their integrals as the sphere.
Sphero-conical coordinates \((\mu_1, \mu_2)\) on \(S^2\)

The parametrization of \((\gamma_1, \gamma_2, \gamma_3) \in S^2\) is

\[
\begin{align*}
\gamma_1^2 &= \frac{(I_1 - \mu_1)(I_1 - \mu_2)}{(I_1 - I_2)(I_1 - I_3)}, \\
\gamma_2^2 &= \frac{(I_2 - \mu_1)(I_2 - \mu_2)}{(I_2 - I_1)(I_2 - I_3)}, \\
\gamma_3^2 &= \frac{(I_3 - \mu_1)(I_3 - \mu_2)}{(I_3 - I_1)(I_3 - I_2)},
\end{align*}
\]

with

\[(\mu_1, \mu_2) \in [I_1, I_2] \times [I_2, I_3].\]
Taking $\mu_1 = \mu_2 = I_2$ one gets four distinguished points in the sphere

$$\left( \pm \sqrt{\frac{I_2 - I_1}{I_3 - I_1}}, 0, \pm \sqrt{\frac{I_3 - I_2}{I_3 - I_1}} \right)$$

that will be paired with the ellipsoid umbilics in the conformal transformation between the surfaces.
From By A.V. Bolsinov, A.T. Fomenko, Integrable Hamiltonian Systems: Geometry, Topology, Classification: The sphero-conical coordinates can be derived from the confocal quadrics coordinates in the limit of very large $x,y,z$.

First, we introduce specific coordinates in $\mathbb{R}^3$, the so-called sphero-conical ones. They can be regarded as a limit case of the elliptic coordinates. To show this, consider the behavior of elliptic coordinates at infinity, more precisely, as $\lambda_1 \to \infty$. This means that the ellipsoids (given as the level surface $\{\lambda_1 = \text{const}\}$) inflate and transform into spheres in the limit. The level surfaces of the second and third families (namely, one-sheeted and two-sheeted hyperboloids) are transformed into two families of elliptic cones at infinity. By applying a contracting homothety to this asymptotic picture, we can transfer the elliptic coordinates “at infinity” into a bounded region in $\mathbb{R}^3$. As a result, we obtain just the sphero-conical coordinates in $\mathbb{R}^3$. The coordinate surfaces of the first family are concentric spheres, and the coordinate surfaces of two others are elliptic cones.
More precisely, the spher-conical coordinates $\nu_2, \nu_3$ are defined to be the roots of the (quadratic) equation

$$\frac{x^2}{a + \nu} + \frac{y^2}{b + \nu} + \frac{z^2}{c + \nu} = 0,$$

which can be thought as the limit of the (cubic) equation

$$\frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda} + \frac{z^2}{c + \lambda} = 1$$

as $(x, y, z) \to \infty$. The first spher-conical coordinate $\nu_1$ is just the sum of the squares:

$$\nu_1 = x^2 + y^2 + z^2.$$

The coordinates $I_1 < \mu_1 < I_2 < \mu_2 < I_3$ cover each octant of the sphere in a similar fashion as the confocal quadric coordinates do for the triaxial ellipsoid.
Proposition 3. The standard metric $d\mathbf{s}^2$ of $S^2$ written in terms of the sphero-conical coordinates:

\[
d\mathbf{s}^2 = \frac{\mu_2 - \mu_1}{4} \left[ \frac{d\mu_1^2}{\prod_{i=1}^{3} (\mu_1 - I_i)} - \frac{d\mu_2^2}{\prod_{i=1}^{3} (\mu_2 - I_i)} \right].
\]
Conformal map from $\mathbb{E}^2(a, b, c)$ to $S^2$

Two conformal maps from a closed simply connected surface to $S^2$ differ by a Moebius transformation.

Such a map will also makes explicit the (unique) complex structure in $\mathbb{E}^2(a, b, c)$, via the global isothermic coordinates $z, \bar{z}$ obtained by stereographic projection of the sphere over the complex plane.

See also a post by Karney, on his cartography forum, suggesting Jacobi + a projection due to Goyou of an hemisphere to asquare in the plane

$S^2$ (round) metric

$$ds^2 = \frac{\mu_2 - \mu_1}{4} \left[ \frac{d\mu_1^2}{\prod_{i=1}^{3} (\mu_2 - I_i)} - \frac{d\mu_2^2}{\prod_{i=1}^{3} (\mu_1 - I_i)} \right].$$

$\mathbb{E}^2$ metric

$$ds^2 = \frac{\lambda_2 - \lambda_1}{4} \left[ \frac{\lambda_1 d\lambda_1^2}{(\lambda_1 - a)(\lambda_1 - b)(\lambda_1 - c)} + \frac{-\lambda_2 d\lambda_2^2}{(\lambda_2 - a)(\lambda_2 - b)(\lambda_2 - c)} \right].$$
FIGURE 3.30. The conformal Guyou projection of the eastern and western hemispheres in squares. This and the Peirce quincuncial (fig. 3.29) are transverse to each other, and either may be indefinitely mosaicked. 10° graticule. Reproduced from Deetz and Adams (1934, 159).
FIGURE 3.29. The conformal Peirce quincuncial projection showing the northern hemisphere in a square and the world in a larger square. The projection involved the first use of elliptic functions in mapping. This map employs the same dividing meridians as those used by Peirce, but with a 15° graticule instead of his 5°.
A simple lemma

**Lemma 1.** Let \((\mu_1, \mu_2) \in T = [a_1, b_1] \times [a_2, b_2]\) and \((\lambda_1, \lambda_2) \in \tilde{T} = [\tilde{a}_1, \tilde{b}_1] \times [\tilde{a}_2, \tilde{b}_2]\) local coordinates on surfaces \(S\) and \(\tilde{S}\), respectively. Assume that the respective metrics can be written as

\[
ds^2(\mu_1, \mu_2) = f(\mu_1, \mu_2) \left[ g_1^2(\mu_1) d\mu_1^2 + g_2^2(\mu_2) d\mu_2^2 \right],
\]

\[
d\tilde{s}^2(\lambda_1, \lambda_2) = \tilde{f}(\lambda_1, \lambda_2) \left[ \tilde{g}_1^2(\lambda_1) d\lambda_1^2 + \tilde{g}_2^2(\lambda_2) d\lambda_2^2 \right]
\]

and that

\[
\int_{a_1}^{b_1} g_1(\mu_1) d\mu_1 = \int_{\tilde{a}_1}^{\tilde{b}_1} \tilde{g}_1(\lambda_1) d\lambda_1 (= r_1), \quad \int_{a_2}^{b_2} g_2(\mu_2) d\mu_2 = \int_{\tilde{a}_2}^{\tilde{b}_2} \tilde{g}_2(\lambda_2) d\lambda_2 (= r_2).
\]

Then the correspondence

\[(\mu_1, \mu_2) \mapsto (\lambda_1(\mu_1), \lambda_2(\mu_2)),\]

defined implicitly through

\[
\int_{a_1}^{\mu_1} g_1(t) dt = \int_{\tilde{a}_1}^{\lambda_1} \tilde{g}_1(t) dt (= \xi_1), \quad \int_{a_2}^{\mu_2} g_2(t) dt = \int_{\tilde{a}_2}^{\lambda_2} \tilde{g}_2(t) dt (= \xi_2).
\]

defines a conformal map between the surfaces.
Conformal factor

\[ d\tilde{s}^2(\mu_1, \mu_2) = h^2(\mu_1, \mu_2) \, ds^2(\mu_1, \mu_2) \]

\[ h^2 = \frac{\tilde{f}(\lambda_1(\mu_1), \lambda_2(\mu_2))}{f(\mu_1, \mu_2)}. \]

In other words, we use the coordinate patch

\((\xi_1, \xi_2) \in R = [0, r_1] \times [0, r_2]\)

as common isothermal parameters for the two surfaces,

\[ ds^2 = f \left[ d\xi_1^2 + d\xi_2^2 \right], \quad d\tilde{s}^2 = \tilde{f} \left[ d\xi_1^2 + d\xi_2^2 \right] \]
We now apply this Lemma for $S^2$ with the spheroconical coordinates

$$(\mu_1, \mu_2)$$

and $E^2$ with Jacobi confocal coordinates

$$(\lambda_1, \lambda_2).$$

Given $a < b < c$ we chose $I_1 < I_2 < I_3$ such that they have the same parametrizing rectangle.
Proposition 6. The conformal factor between the metric on the ellipsoid to the metric on the sphere is

\[ h^2 = \frac{\lambda_2(\mu_2) - \lambda_1(\mu_1)}{\mu_2 - \mu_1}. \]

The functions \( \lambda_2(\mu_2), \lambda_1(\mu_1) \) defining the conformal map are derived from the metric expressions in the previous slide.

Identifying the corresponding points of the two surfaces on the double coverings by the common lattice in the complex plane \( w = u + iv \), a global map from \( \mathbb{E}^2 \) to \( S^2 \) results.

\( w, \bar{w} \) are common local isothermal coordinates. The umbilical points in the ellipsoid (and the special points in the sphere) are ramification points of order 1/2.
Technical details

We consider $a \neq 0$.

One should consider separately the case $a = 0$ (double faced elliptical region) for which it is simpler.

Let $F$ the Legendre elliptic integral of the first kind

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^t \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}$$

where $t = \sin \phi$ and $\Pi$ of the third kind

$$\Pi(\phi, \ell, n) = \int_0^\phi \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{1 - \ell^2 \sin^2 \theta}}.$$
The relations between $\lambda_i$ and $\mu_i, i = 1, 2$ are given by

$$P(\lambda_1) = S(\mu_1), \quad Q(\lambda_2) = T(\mu_2)$$

$$P(\lambda_1) = \int_a^{\lambda_1} \sqrt{\frac{t}{(t-a)(t-b)(t-c)}} \, dt = \frac{2a}{\sqrt{b(c-a)}} \Pi(\phi_1, \ell_1, n_1)$$

with

$$\phi_1 = \arcsin \sqrt{\frac{b(\lambda_1 - a)}{\lambda_1(b-a)}},$$

$$\ell_1 = \sqrt{\frac{1 - (a/b)}{1 - (a/c)}} (0 < \ell_1 < 1) \quad \text{and} \quad -1 < n_1 = -1 + a/b < 0.$$ 

$$Q(\lambda_2) = \int_{\lambda_2}^{c} \sqrt{-\frac{-t}{(t-a)(t-b)(t-c)}} \, dt = \frac{2c}{\sqrt{b(c-a)}} \Pi(\phi_2, \ell_2, n_2)$$

with

$$\phi_2 = \arcsin \sqrt{\frac{b(c-\lambda_2)}{\lambda_2(c-b)}}, \quad \ell_2 = \sqrt{\frac{(c/b) - 1}{(c/a) - 1}} (0 < \ell_2 < 1) \quad \text{and} \quad n_2 = c/b - 1 > 0.$$
\[ S(\mu_1) = \int_{I_1}^{\mu_1} \sqrt{\frac{1}{(t - I_1)(t - I_2)(t - I_3)}} \, dt = \frac{2}{\sqrt{I_3 - I_1}} F(\phi_1, k_1) \]

\[ \phi_1 = \arcsin \sqrt{\frac{\mu_1 - I_1}{I_2 - I_1}}, \quad k_1 = \sqrt{\frac{I_2 - I_1}{I_3 - I_1}}; \]

\[ T(\mu_2) = \int_{\mu_2}^{I_3} \sqrt{\frac{-1}{(t - I_1)(t - I_2)(t - I_3)}} \, dt = \frac{2}{\sqrt{I_3 - I_1}} F(\phi_2, k_2) \]

\[ \phi_2 = \arcsin \sqrt{\frac{I_3 - \mu_2}{I_3 - I_2}}, \quad k_2 = \sqrt{\frac{I_3 - I_2}{I_3 - I_1}}. \]
**Master equation: the relation between** $(a, b, c)$ **and** $(I_1, I_2, I_3)$

Both $k_1, k_2$ and $\ell_1, \ell_2$ are complementary: $k_1^2 + k_2^2 = 1, \ell_1^2 + \ell_2^2 = 1$.

We denote

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad 0 \leq k \leq 1$$

the complete integral of the first type.

We now fulfill the requirement that the umbilical points of the ellipsoid should correspond to the special points in the sphere so that the map between $E^2(a, b, c)$ and $S^2$ is well defined. This amounts the two equalities

$$K(k_1) = \frac{\sqrt{I_3 - I_1}}{2} P(b), \quad K(k_2) = \frac{\sqrt{I_3 - I_1}}{2} Q(b)$$

where we have the complete integrals of the third kind

$$P(b) = \frac{2a}{\sqrt{b(c - a)}} \Pi(\pi/2, \ell_1, n_1)$$

$$Q(b) = \frac{2c}{\sqrt{b(c - a)}} \Pi(\pi/2, \ell_2, n_2).$$
Proposition 7. *(Master Equation)*

\[
\frac{K\left(\sqrt{1-k_1^2}\right)}{K(k_1)} = \frac{Q(b)}{P(b)} = \frac{c}{a} \frac{\prod(\pi/2, \ell_2, n_2)}{\prod(\pi/2, \ell_1, n_1)}
\]

After getting \(k_1\), the parameters \(I_1, I_2\) and \(I_3\) are obtained from

\[
I_3 - I_1 = 4 \left(\frac{K(k_1)}{P(b)}\right)^2 = 4 \left(\frac{K(k_2)}{Q(b)}\right)^2 \tag{2}
\]

\[
I_2 - I_1 = (I_3 - I_1) k_1^2, \quad (I_3 - I_2 = (I_3 - I_1) k_2^2 \text{ is redundant})
\]

*No harm is done by setting \(I_1 = 0\) for simplicity.*

The left hand side of the Master Equation decreases from \(\infty\) to 0 as \(k_1\) runs from 0 to 1. Thus there is an unique solution to this equation.

The special points in the sphere corresponding to the umbilics are

\((\pm k_1, 0, \pm k_2)\).
About the equation \( K \left( \sqrt{1 - k^2} \right) / K(k) = \sqrt{r} \)

The solution \( k = \Lambda(r) \) of this equation is called the *elliptic lambda function* and can be obtained via Jacobi theta functions

\[
k = \left[ \frac{\theta_2(0, q_r)}{\theta_3(0, q_r)} \right]^2
\]

\( q_r = \exp(-\pi/\sqrt{r}) \) is called the *nome* and the theta functions are

\[
\theta_2(0, q) = \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2}, \quad \theta_3(0, q) = \sum_{m=-\infty}^{\infty} q^{m^2}.
\]
Historical note

Riemann began his study of the quotient \( \frac{K'}{K} \) by determining how \( K \) and \( K' \), when regarded as functions of \( k^2 \), would change as \( k^{-2} \) made a circuit around the branch points 0, 1, \( \infty \). For this purpose, he took the hypergeometric functions \( K \) and \( K' \) to be defined by the integrals

\[
K = \int_0^1 (1 - t^2)^{-\frac{1}{2}} (1 - k^2 t^2)^{-\frac{1}{2}} \, dt = \int_0^1 \frac{1}{2} x^{-\frac{1}{2}} (1 - x)^{-\frac{1}{2}} (1 - k^2 x)^{-\frac{1}{2}} \, dx \quad (7.75)
\]

and

\[
K' = \int_0^1 (1 - t^2)^{-\frac{1}{2}} (1 - k^2 t^2)^{-\frac{1}{2}} \, dt = -i \int_1^{k^2} \frac{1}{2} x^{-\frac{1}{2}} (1 - x)^{-\frac{1}{2}} (1 - k^2 x)^{-\frac{1}{2}} \, dx.
\quad (7.76)
\]
We see from (7.74) that if we set $\tau = i \frac{K'}{K}$, then the change in $\frac{K'}{K}$ referred to by Jacobi may be written $\tau' = \frac{b+b' \tau}{a+a' \tau}$. Thus, as early as 1828, Jacobi was aware that $k^2$ was a modular function of $\frac{iK'}{K}$ with respect to the subgroup $\Gamma(2)$ of the full modular group. In his lectures on hypergeometric functions, Riemann proved Jacobi’s theorem that $k^2$ was invariant with respect to $\Gamma(2)$. In the course of the proof, he found the generators of the group $\Gamma(2)$ and constructed its fundamental domain. He also considered the inverse case, $\frac{K'}{K}$ as a function of $k^2$; he demonstrated that this function was a conformal mapping from the upper half-plane to half of the fundamental domain of $\Gamma(2)$. Note
27. 

Suite des notices sur les fonctions elliptiques. (V. p. 192.)

(Par Mr. C. G. J. Jacobi, prof. en phil. à Königsberg.)

(Extrait d'une lettre de l'auteur au redacteur de ce journal, du 21. Juillet 1828.)

J'ajoute aux formules données dans ma dernière lettre les développements des fonctions elliptiques de seconde et de troisième espèce.

Soit en adoptant la notation de Mr. Legendre, \( \triangle = \sqrt{1 - k^2 \sin \varphi} \),

\[
E \phi = \int_0^\varphi \triangle \, \partial \phi, \quad E^1 = \int_0^{\pi/2} \triangle \, \partial \phi, \quad F \phi = \int_0^\varphi \frac{\partial \phi}{\triangle}, \quad F^1 = \int_0^{\pi/2} \frac{\partial \phi}{\triangle},
\]

de manière que \( F^1 \) est ce que je désigne par \( K \): si l'on met

\[
\Phi = a \, m \, \frac{2Kx}{K}, \quad q = e^{-Kx/K},
\]
Les fonctions elliptiques diffèrent essentiellement des transcendantes ordinaires. Elles ont une manière d'être pour ainsi dire absolue. Leur caractère principal est d'embrasser tout ce qu'il y a de périodique dans l'analyse. En effet les fonctions trigonométriques ayant une période réelle,

Crelle's Journal. III. Bd. 3. Hft.


les exponentielles une période imaginaire, les fonctions elliptiques embrassent les deux cas, puisqu'on a en même temps

\[ \sin am(u + 4K) = \sin am(u), \]
\[ \sin am(u + 2iK') = \sin am(u), \]

\( i \) étant \( = \sqrt{-1} \). D'ailleurs on démontre aisément qu'une fonction analytique ne saura avoir plus que deux périodes, l'une réelle et l'autre imaginaire ou l'une et l'autre imaginaires. Ce dernier cas répond à un module \( k \) imaginaire. Le quotient \( \frac{K'}{K} \) des deux périodes d'une fonction proposée détermine le module \( k \) des fonctions elliptiques par lesquelles elle doit être exprimée au moyen des formules 15., 17. Il conviendra peut-être à introduire dans l'analyse des fonctions elliptiques ce quotient \( \frac{K'}{K} \) comme module au lieu de \( k \). A l'égard de ce quotient j'ai trouvé
"que \( k \) ne change pas de valeur, si l'on écrit au lieu de \( \frac{K'}{K} \) l'expression
\[
\frac{bK + i b'K'}{aK + i a'K'} = \frac{KK' - i(abKK + a'b'K'K')}{a a KK + a'a'K'K'},
\]
"\( a, a', b, b' \) étant des nombres entiers quelconques, \( a \) un nombre impair, \( b \) un nombre pair, tels que \( ab' - a'b = 1; \)" 
théorème remarquable et qui doit être envisagé comme un des théorèmes fondamentaux de l'analyse des fonctions elliptiques.

Les méthodes qui m'ont conduit à la théorie générale de la transformation des fonctions elliptiques s'appliquent également à une classe très étendue d'intégrales doubles, triples, et même d'intégrales multiples d'un ordre quelconque. Un premier essai sur cette matière épineuse a été donné dans un petit mémoire qui a pour titre:

"De singulari quadam duplicis integralis transformatione"
inseré au second volume de votre journal.

Vous voyez, Monsieur, que la théorie des fonctions elliptiques est un vaste objet de recherches qui dans le cours de ses développements embrassent presque toute l'algèbre, la théorie des intégrales définies et la science des nombres. Quel titre de gloire pour l'illustre auteur du traité des fonctions elliptiques, que d'avoir créé cette belle théorie et d'avoir allumé ce flambeau à la posterité.
Global map from $\mathbb{E}^2$ to $\mathbb{S}^2$

We identify the corresponding points on the two surfaces (ellipsoid and the sphere) on their double coverings by the same lattice in the complex $w$-plane ($w = u + iv$).

**Proposition 4.** Global isothermal coordinates $(z, \bar{z})$ on $\mathbb{E}^2$ (except for the point corresponding to $z = \infty$) are obtained by stereographic projection from the $\gamma$-sphere to the complex $z$-plane, namely:

$$w = u + iv \text{ (lattice)} \rightarrow (p \in \mathbb{E}^2) \leftrightarrow (\gamma \in \mathbb{S}^2) \rightarrow z \in \mathbb{C}.$$  

$z = z(w)$ is a (complex) elliptic function.
References


Vortices on Closed Surfaces, with S. Boatto