Asymptotic Poincaré Maps along the edges of Polytopes and their Hamiltonian character

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- Asymptotic Poincaré Map
- Hamiltonian Polymatirx Replicators
- Hamiltonian character of asymptotic poincaé map.
- Examples

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Definition

A set $\Gamma^d \subset \mathbb{R}^N$ is called a *d*-dimensional simple polytope if it is a compact convex subset of dimension *d*, with affine support $E^d \subset \mathbb{R}^N$, for which there exist a family of affine functions $\{f_i : E^d \to \mathbb{R}\}_{i\in I}$ such that

(a) $\Gamma^d = \cap_{i\in I} f_i^{-1}([0, +\infty[)$.

(b) $\Gamma^d \cap f_i^{-1}(0) \neq \emptyset \qquad \forall i \in I$.

(c) Given $J \subseteq I$ such that $\Gamma^d \cap (\cap_{j\in J} f_j^{-1}(0)) \neq \emptyset$, the linear 1-forms $\mathrm{d}f_j$ are linearly independent at every point $p \in \cap_{j\in J} f_j^{-1}(0)$.

- V set vertices v,
- E set of edges γ and E_V set of edges containing v.
- F set of faces σ and F_v set of faces containing v.
- the set of corners
  $C := \{ (v, \gamma, \sigma) \in V \times E \times F : \gamma \cap \sigma = \{v\} \}$.

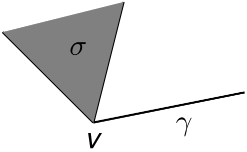
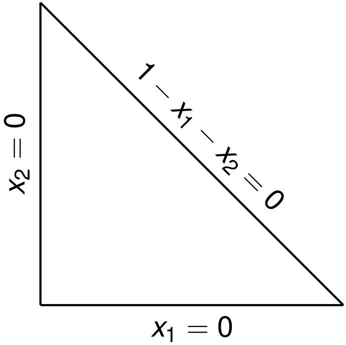


Figure: A corner (v, γ, σ) in a three dimensional polytope.

By (c) of Definition 1 at any given vertex v, the co-vectors $(df_i)_v$ are linearly independent. So in a small enough neighborhood of v the functions $\{f_\sigma\}_{\sigma \in F_v}$ can be used as a system of coordinates.

Example: $\Gamma^2 \subset \mathbb{R}^2$ defined by

$$f_1(x) = x_1, \quad f_2(x) = x_2, \quad f_3(x) = 1 - x_1 - x_2$$



$\mathcal{A}(\Gamma^d) :=$ the space of functions defined on Γ^d that can be extended analytically to a neighborhood of Γ^d,

$\mathfrak{X}(\Gamma^d) :=$ the space of vector fields X, defined on Γ^d, that have an analytic extension to a neighborhood of Γ^d and such that X is tangent to every r-dimensional face of Γ^d, for all $0 \leq r \leq d$.

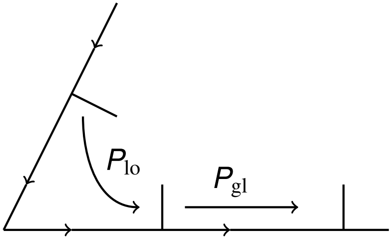


Figure: The local, P_{lo}, and global, P_{gl}, Poincaré maps along a heteroclinic orbit.

The asymptotic linearization of P_{gl} will be identity.

$$N_v := \{p \in \Gamma^d : 0 \leq x_j(p) \leq 1 \text{ for } 1 \leq j \leq d\}$$

where $(x_1, \ldots, x_d)$ is a system of affine coordinates around v which assigns coordinates $(0, \ldots, 0)$ to v and such that the hyperplanes $x_j = 0$ are precisely the facets of the polytope through v.

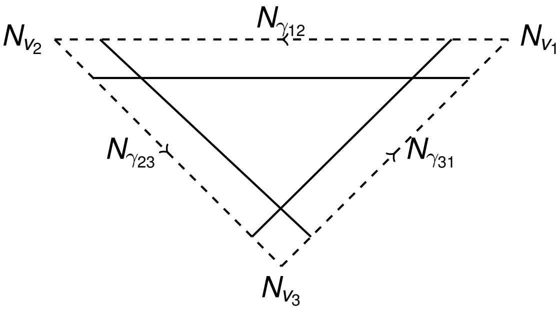


Figure: Tubular neighborhoods along the edges in a two dimensional polytope

We blow up this tubular neighborhood by a rescalling change of coordinates. The rescaling change of coordinates Ψ_ϵ^X takes points in N_v to points in the sector

$$\Pi_v := \{(y_\sigma)_{\sigma \in F} \in \mathbb{R}_+^{|F|} \text{ s.t. } y_\sigma = 0 \text{ when } \sigma \notin F_v\}.$$

If $F_v = \{\sigma_1, \dots, \sigma_d\}$, the map Ψ_ϵ^X, generically, is defined on the neighborhood N_v by

$$\Psi_\epsilon^X(q) := (-\epsilon^2 \log x_1(q), \dots, -\epsilon^2 \log x_d(q), 0, \dots, 0).$$

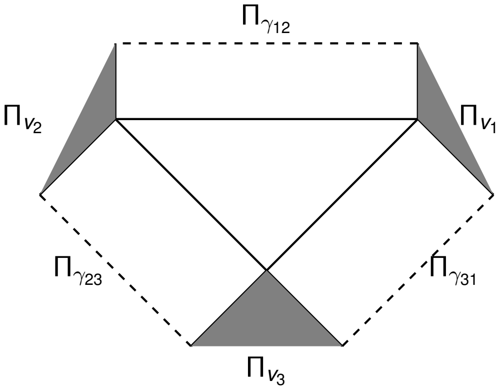


Figure: Range of the rescaling change of coordinates in the dual cone

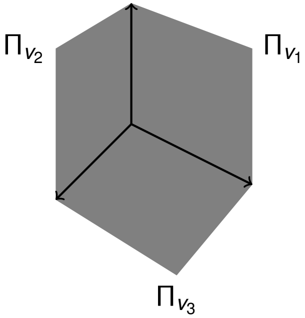


Figure: Dual cone of a triangle in $\mathbb{R}^{|F|}$

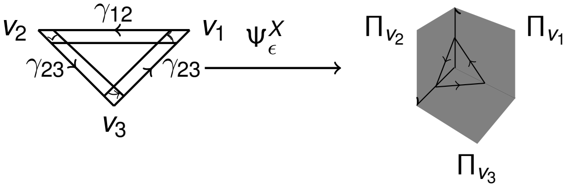
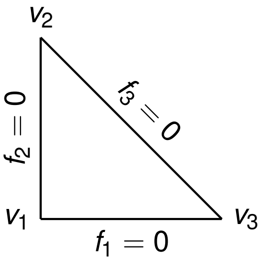
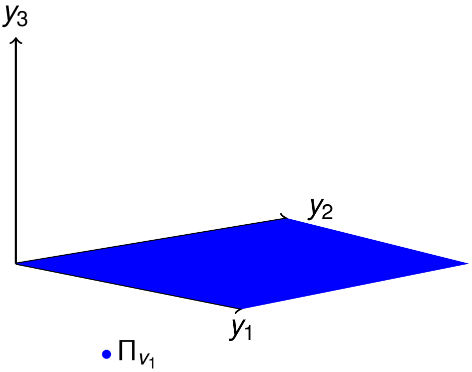
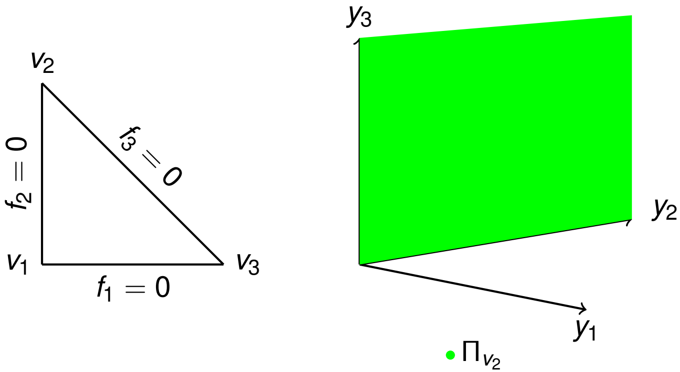
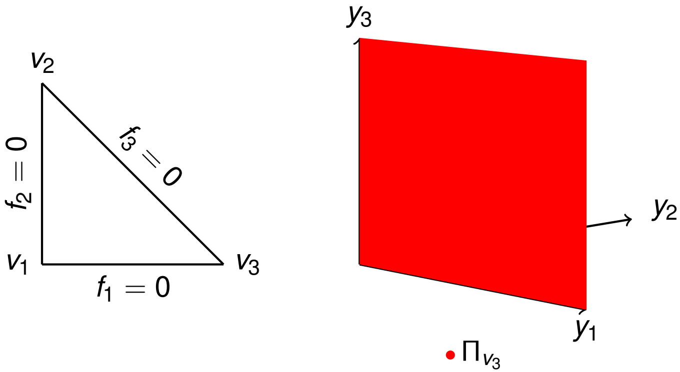


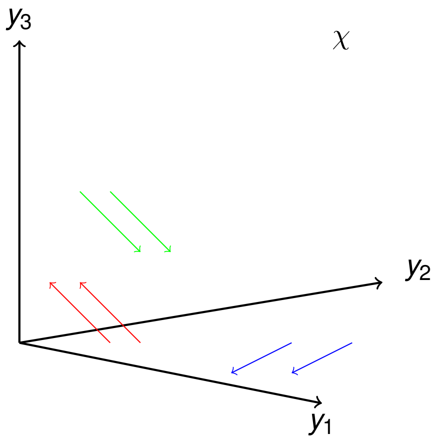
Figure: Asymptotic linearization


v_2
$f_2 = 0$
$f_3 = 0$
v_1
v_3
$f_1 = 0$


y_3
y_2
y_1
Π_{v_1}


v_2
$f_2 = 0$
$f_3 = 0$
v_1
v_3
$f_1 = 0$
y_3
y_2
y_1
Π_{v_2}


v_2
$f_2 = 0$
$f_3 = 0$
v_1
v_3
$f_1 = 0$
y_3
y_2
y_1
Π_{v_3}


y_3
χ
y_2
y_1

$X \in \mathfrak{X}(\Gamma^d)$ is tangent to a given face $\sigma := \{q \in \Gamma^d, f(q) = 0\}$, then

$$df_\sigma(X) = 0 \Rightarrow \exists H_\sigma \in \mathcal{A}(\Gamma^d) \text{ s.t. } df_\sigma(X) = (F_\sigma)^{\nu(X,\sigma)} H_\sigma$$

For a smooth vector field X, the order $\nu(X, v, \sigma)$ at a corner (v, σ) is the minimum integer $k \geq 1$ such that $(\mathrm{d}f_\sigma)_v(D^k X)_v \neq 0$. The order of σ is defined as

$$\nu(X, \sigma) := \min_v \{\nu(X, v, \sigma) : \sigma \in F_v\} .$$

Order function: $\nu : \mathfrak{X}(\Gamma^d) \times F \to \{1, 2, 3, \ldots\}$.

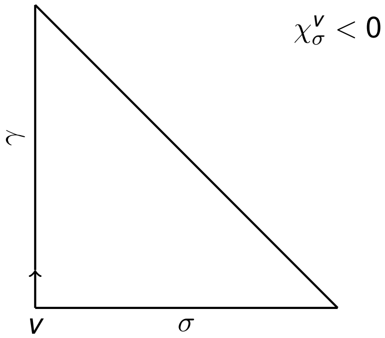
Definition

The *skeleton character* of $X \in \mathfrak{X}(\Gamma^d)$ is defined to be the family $\chi := (\chi_\sigma^v)_{(v,\sigma)\in V\times F}$ where

$$\chi_\sigma^v := \begin{cases} -H_\sigma(v) & \sigma \in F_v \\ 0 & \text{otherwise} \end{cases},$$

while the *skeleton character* at $v \in V$, is $\chi^v := (\chi_\sigma^v)_{\sigma\in F}$.

Consider a corner (v, γ, σ), if $\nu(X, \sigma) = 1$ then χ_σ^v is simply minus the eigenvalue of the linearization of X at equilibrium v along its eigenvector γ.



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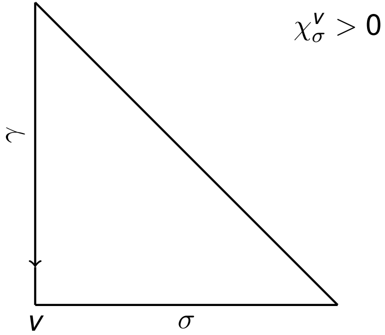


Note that χ^v is tangent to
$\Pi_v := \{(y_\sigma)_{\sigma\in F} \in \mathbb{R}_+^{|F|}$ s.t. $y_\sigma = 0$ when $\sigma \notin F_v\}$.

Definition

- The dual cone of the polytope Γ^d is $\mathcal{C}^*(\Gamma^d) := \cup_{v\in V}\Pi_v \subset \mathbb{R}^{|F|}$.
- Given $X \in \mathfrak{X}(\Gamma^d)$, the family $\chi := \{\chi_v\}_{v\in V}$ is called the skeleton vector field associated to X.

For every $n = 1, 2, \ldots$, define $h_n : (0, \infty) \to \mathbb{R}$ to be

$$h_1(x) = -log\, x \quad \text{and} \quad h_n(x) = -\frac{1}{n-1}\left(1 - \frac{1}{x^{n-1}}\right) \quad n \geq 2. \tag{1}$$

Remark

This family can also be characterized by the properties: $h'_n(x) = -x^{-n}$, $h_n(0) = +\infty$ and $h_n(1) = 0$, which prove that the function $h_n :]0, 1] \to [0, +\infty[$ is a diffeomorphism.

Definition

Given a vector field $X \in \mathfrak{X}(\Gamma^d)$ with $n = |F|$ facets we define the ϵ-rescaling coordinate system $\Psi_\epsilon^X : N_{\Gamma^d} \setminus \partial\Gamma^d \to \mathbb{R}^n$ which maps $q \in N_{\Gamma^d}$ to $y := (y_{\sigma_1}, \ldots, y_{\sigma_n})$ where

- If $q \in N_v$ for some vertex v:

$$y_{\sigma_i} = \begin{cases} \epsilon^2 h_{\nu(X,\sigma_i)}(f_\sigma(q)) & \text{if} \quad \sigma \in F_v \\ 0 & \text{if} \quad \sigma \notin F_v \end{cases}$$

- If $q \in N_\gamma$ for some edge γ:

$$y_{\sigma_i} = \begin{cases} \epsilon^2 h_{\nu(X,\sigma_i)}(f_\sigma(q)) & \text{if} \quad \gamma \subset \sigma \\ 0 & \text{if} \quad \gamma \not\subset \sigma \end{cases}$$

Definition

Suppose we are given a family of functions, or mappings, F_ϵ with varying domains $\mathcal{U}_\epsilon$. Let F be another function with domain $\mathcal{U}$. Assume that all these functions have the same target and source spaces, which are assumed to be linear spaces. We will say that $\lim_{\epsilon\to 0^+} F_\epsilon = F$ *in the* C^∞ *topology*, to mean that:

1. *domain convergence*: for every compact subset $K \subseteq \mathcal{U}$, we have $K \subseteq \mathcal{U}_\epsilon$ for all small enough $\epsilon > 0$, and
2. *derivative uniform convergence on compacts*: for every $k \in \mathbb{N}$

$$\lim_{\epsilon\to 0^+} \sup_{u\in K\ 0\leq i\leq k} \left| D^i \left[F_\epsilon(u) - F(u)\right] \right| = 0 .$$

For a given vertex $v \in V$ we define

$$\Pi_v(\varepsilon) := \{\, y \in \Pi_v : \; y_\sigma \geq \varepsilon \quad \text{for all} \quad \sigma \in F_v \,\} \tag{2}$$

Lemma

The push-forward of X by $\Psi^X_{v,\varepsilon}$ is

$$(\Psi^X_{v,\varepsilon})_* X = \varepsilon^2 \tilde{X}^\varepsilon_v,$$

where $\tilde{X}^\varepsilon_v := \left(-H_\sigma\left((\Psi^X_{v,\varepsilon})^{-1}(y)\right)\right)_{\sigma \in F_v}$. Furthermore, the following limit holds in the C^∞ topology

$$\lim_{\varepsilon \to 0} (\tilde{X}^\varepsilon_v)_{|\Pi_v(\varepsilon)} = \chi^v \,.$$

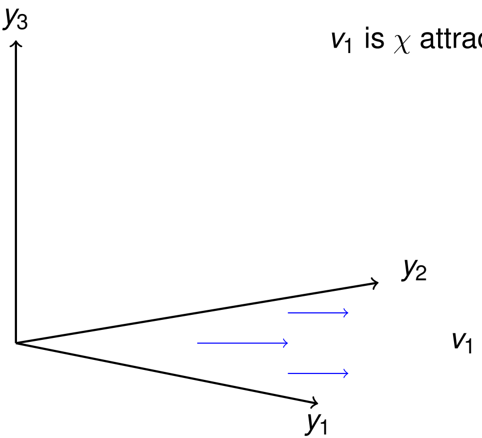
Let $X \in \mathfrak{X}(\Gamma^n)$ and χ be its skeleton vector field.

Definition

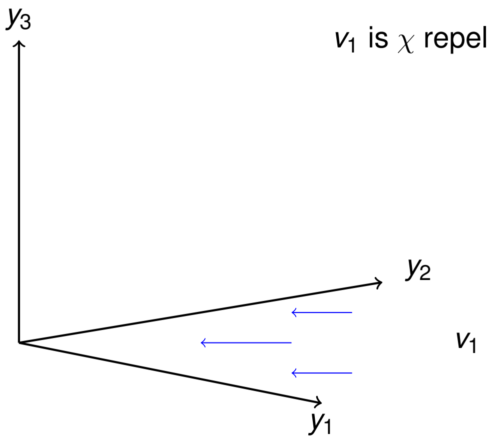
The vertex v is called

- χ-*attractive* if $\chi^v \in \Pi_v$
- χ-*repelling* if $-\chi^v \in \Pi_v$
- of *saddle type* otherwise.

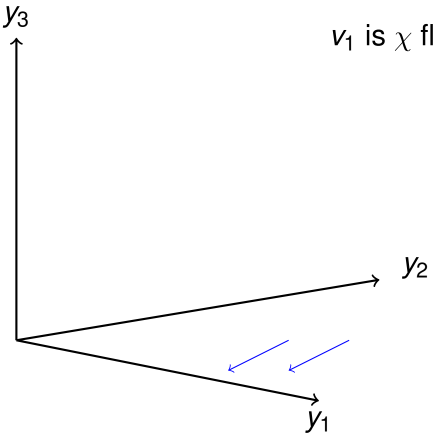
v_1 is χ attractive



v_1 is χ repelling



v_1 is χ flowing

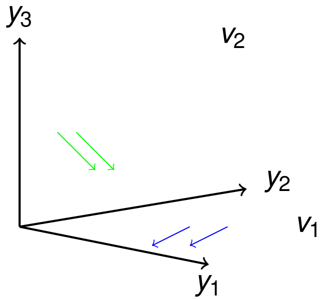


Definition

The edge γ is with end corners (v_i, σ_i) and (v_j, σ_j) is called χ-*defined* if either $\chi^{v_i}_{\sigma_i} \chi^{v_j}_{\sigma_j} \neq 0$ or else $\chi^{v_i}_{\sigma_i} = \chi^{v_j}_{\sigma_j} = 0$. Moreover, we say that γ is

- a *flowing-edge* if $\chi^{v_i}_{\sigma_i} \chi^{v_j}_{\sigma_j} < 0$,
- a *neutral edge* if $\chi^{v_i}_{\sigma_i} = \chi^{v_j}_{\sigma_i} = 0$,
- an *attracting edge* if $\chi^{v_i}_{\sigma_i} < 0$ and $\chi^{v_j}_{\sigma_j} < 0$,
- a *repelling edge* if $\chi^{v_i}_{\sigma_i} > 0$ and $\chi^{v_j}_{\sigma_j} > 0$,
- a χ-*undefined* if non of the above happens.

For flowing-edges we write $v_i \xrightarrow{\gamma} v_j$ whenever $\chi^{v_i}_{\sigma_i} < 0$ and $\chi^{v_j}_{\sigma_j} > 0$. The vertex $v_i := s(\gamma)$ called source and $v_j := t(\gamma)$ called target.

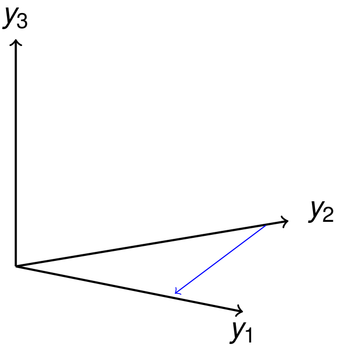


The flowing edge $v_2 \xrightarrow{\gamma} v_1$

For a given vertex v_i of saddle type together with an incoming flowing-edge $(v_{i-1}, \sigma_j^s) \xrightarrow{\gamma_j} (v_i, \sigma_j^t)$ and an outgoing flowing-edges $(v_i, \sigma_{j+1}^s) \xrightarrow{\gamma_{j+1}} (v_{i+1}, \sigma_{j+1}^t)$, let

$$L_{\gamma_j,\gamma_{j+1}}(y) := \left(y_\sigma - \frac{\chi_\sigma^{v_i}}{\chi_{\sigma_{j+1}^s}^{v_i}} \, y_{\sigma_{j+1}^s} \right)_{\sigma \in F} \tag{3}$$

with the domain $\pi_{\gamma_j,\gamma_{j+1}} := \{ y \mid y_\sigma - \frac{\chi_\sigma^{v_i}}{\chi_{\sigma_{j+1}^s}^{v_i}} \, y_{\sigma_{j+1}^s} > 0 \}$.



v_1

$L^{\chi}_{\gamma,\gamma'}(x)$

Definition

A sequence of edges $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$ is called a χ-*path* if

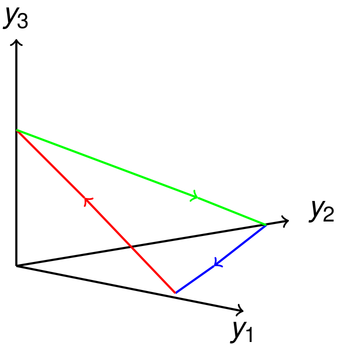
1. γ_j is flowing, for all $j = 0, 1, \ldots, m$,
2. $t(\gamma_{j-1}) = s(\gamma_j)$, for all $j = 1, \ldots, m$.

The χ-path ξ is called a *cycle* when $\gamma_0 = \gamma_m$. The integer m is called the *length* of the path.

Definition

Given a χ-path $\xi = (\gamma_0, \gamma_1, \ldots, \gamma_m)$, we define the *skeleton Poincaré map of* χ *along* ξ to be the mapping $\pi_\xi : \Pi_\xi \to \Pi_{\gamma_m}$

$$\pi_\xi := L_{\gamma_{m-1},\gamma_m} \circ \ldots \circ L_{\gamma_0,\gamma_1} \ ,$$


y_3
y_2
y_1

Theorem

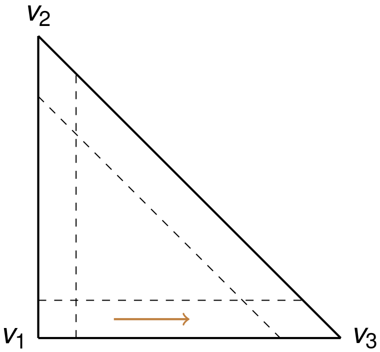
Given a flowing-edge $(v_i, \sigma^s_{j+1}) \xrightarrow{\gamma_{j+1}} (v_{i+1}, \sigma^t_{j+1})$, *let* $\mathcal{U}^\epsilon_{\gamma_{j+1}} \subset \Pi_{\gamma_{j+1}}(\epsilon^r)$ *be the domain of the map*

$$F^\epsilon_{\gamma_{j+1}} := \Psi^X_{v_{i+1},\epsilon} \circ P_{\gamma_{j+1}} \circ (\Psi^X_{v_i,\epsilon})^{-1},$$

where $P_{\gamma_{j+1}}$ *is the poincare map along* γ_{j+1}. *Then*

$$\lim_{\epsilon \to 0^+} F^\epsilon_{\gamma_{j+1}}|_{\mathcal{U}^\epsilon_{\gamma_{j+1}}} = I$$

in the C^k *topology, in the sense of Definition 5.*

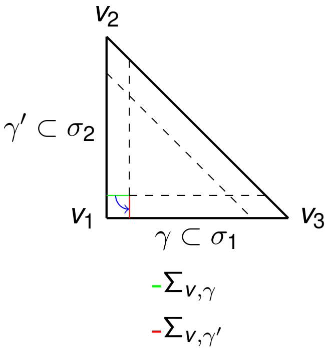

v_2
v_1
v_3

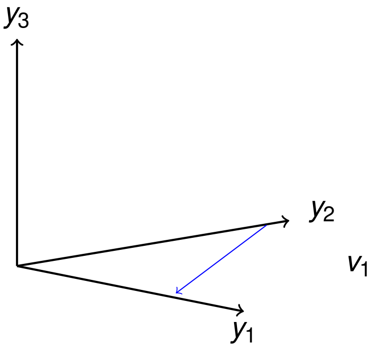
Theorem

Given $\gamma_j, \gamma_{j+1} \in E_\chi$*, let* $\mathcal{U}^{\epsilon}_{\gamma_j,\gamma_{j+1}} \subset \Pi_{\gamma_j}(\epsilon^r)$ *(r taken according to Lemma 6) be the domain of the map* $F^{\epsilon}_{\gamma_j,\gamma_{j+1}} := \Psi^X_{v_i,\epsilon} \circ P_{\gamma_j,\gamma_{j+1}} \circ (\Psi^X_{v_i,\epsilon})^{-1}$*. Then*

$$\lim_{\epsilon \to 0^+} \left(F^{\epsilon}_{\gamma_j,\gamma_{j+1}} \right)_{|\mathcal{U}^{\epsilon}_{\gamma_j,\gamma_{j+1}}} = (L_{\gamma_j,\gamma_{j+1}})_{|\Pi_{\gamma_j,\gamma_{j+1}}}$$

in the C^k *topology, in the sense of Definition 5.*


v_2
$\gamma' \subset \sigma_2$
v_1
v_3
$\gamma \subset \sigma_1$
$-\Sigma_{v,\gamma}$
$-\Sigma_{v,\gamma'}$


y_3
y_2
v_1
y_1

Evolutionary Game Theory

EGT originated from the work of:

John Maynard Smith and George R. Price

who applied the theory of strategic games developed by :

John von Neumann and Oskar Morgenstern

to evolution problems in Biology.

Unlike Game theory which is more concentrated on mathematical modeling, EGT investigates the dynamical processes of biological populations.

Polymatrix Replicator Equation

Let $\underline{n} = (n_1, \ldots, n_p)$ where p and n_i are positive integers and set $n = n_1 = \ldots + n_p$. We denote

- $\mathbb{R}^n \ni x = (x^\alpha)_\alpha$, where $x^\alpha \in \mathbb{R}^{n_\alpha}$, $\alpha = 1, \ldots, p$.
- $i \in \alpha$, if and only if $n_1 + \ldots + n_{\alpha-1} < i \leq n_1 + \ldots + n_\alpha$.

Consider a block matrix

$$A = \left[A^{\alpha,\beta} \right]_{\alpha,\beta},$$

where $\alpha, \beta = 1, \ldots, p$ and each block $A^{\alpha,\beta} = \left[a_{ij}^{\alpha,\beta} \right]_{ij}$ is a $n_\alpha \times n_\beta$ real matrix.

We define the polymatrix replicator equation associated to A, on the prism

$$\Gamma_{\underline{n}} = \Delta^{n_1 - 1} \times \ldots \times \Delta^{n_p - 1},$$

where $\Delta^{n_\alpha - 1} = \{x \in \mathbb{R}_+^{n_\alpha} \mid \sum_{i=1}^{n_\alpha} x_i = 1\}$, by

$$\dot{x}^{\alpha_i} = x_i^\alpha \left((A\,x)_i - \sum_{\beta=1}^{p} (x^\alpha)^t A^{\alpha,\beta} x^\beta \right) \quad \forall\, i \in \alpha,\ \alpha \in \{1, \ldots, p\}\ , \tag{4}$$

We will denote $X_{(\underline{n},A)} = (\dot{x}^1, \ldots, \dot{x}^n)$.

Remark

$X_{(\underline{n},A)}$ preserves $\Gamma_{\underline{n}}$ and all its faces. Restriction of $X_{(\underline{n},A)}$ to any face is also a Polymatrix Replicator equation.

Polymatrix replicator equation is a mathematical model for polymatrix games and generalizes

1) Replicator equations when $p = 1$,
2) Asymetric Replicator equations (Bimatrix Games) when $p = 2$ and

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

Proposition (Equilibria)

A point $q \in \Gamma_{\underline{n}}$ is an equilibrium of $X_{(\underline{n},A)}$ if $(Aq)_i = (Aq)_j$, for all $\alpha = 1, \ldots, p$ and every $i, j \in \alpha$.
Moreover, if $q \in \Gamma^{\circ}_{\underline{n}}$ is an equilibrium point then $(Aq)_i = (Aq)_j$, for all $\alpha = 1, \ldots, p$ and every $i, j \in \alpha$.

Definition

A *formal equilibrium* of a polymatrix game $G = (\underline{n}, A)$ is any vector $q \in \mathbb{R}^n$ such that

(a) $(A\,q)_i = (A\,q)_j$ for all $i, j \in \alpha$, and all $\alpha = 1, \ldots, p$,
(b) $\sum_{j\in\alpha} q_j = 1$ for all $\alpha = 1, \ldots, p$.

Hamiltonian Replicator Equations

Notation:

$$\mathbb{1}_n := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

and given $x \in \mathbb{R}^n$, we denote by D_x the $n \times n$ diagonal matrix $D_x = \mathrm{diag}(x_i)_i$.
For each $\alpha \in \{1, \ldots, p\}$ we define the $n_\alpha \times n_\alpha$ matrix

$$T_x^\alpha := x^\alpha \, \mathbb{1}^t - I \; ,$$

and set T_x to be the $n \times n$ block diagonal matrix $T_x = \mathrm{diag}(T_x^\alpha)_\alpha$.

Definition

Given a polymatrix game $G = (\underline{n}, A)$, we define the matrix valued mapping $\pi_A : \mathbb{R}^n \to \mathrm{Mat}_{n\times n}(\mathbb{R})$

$$\pi_A(x) := (-1)\, T_x\, D_x\, A\, D_x\, T_x^t. \tag{5}$$

Notice that $\pi_A(x)$ is a skew symmetric matrix valued map whenever A is a skew symmetric matrix.

Proposition

Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, assume there exists a formal equilibrium $q \in \mathbb{R}^n$ of $G = (\underline{n}, A)$. Then, setting $H(x) = \sum_{i=1}^n q_i \log x_i$,

$$X_{(\underline{n},A)}(x) = \pi_A(x)\, \mathrm{d}_x H \quad \text{for every} \quad x \in \Gamma_{\underline{n}} \ .$$

In fact the above equation holds for every point in the affine space generated by $\Gamma_{\underline{n}}$.

Theorem

When A is anti-symmetric, π_A defines a Poisson structure on $\Gamma_{\underline{n}}$.

Proof

We define diffeomorphism $\phi : \mathbb{R}^{n-p} \to \Gamma^{\circ}_{\underline{n}}$, where $\phi(u^\alpha)_\alpha := (\phi^\alpha(u^\alpha))_\alpha$ for $\alpha = 1, .., p$ and each component $\phi^\alpha : \mathbb{R}^{n_\alpha - 1} \to (\Delta^{n_\alpha - 1})^\circ$ is the map defined by

$$\phi^\alpha(u^\alpha) := \left(\frac{e^{u_1^\alpha}}{1 + \sum_{i=1}^{n_\alpha - 1} e^{u_i^\alpha}}, \ldots, \frac{e^{u_{n_\alpha - 1}^\alpha}}{1 + \sum_{i=1}^{n_\alpha - 1} e^{u_i^\alpha}}, \frac{1}{1 + \sum_{i=1}^{n_\alpha - 1} e^{u_i^\alpha}} \right) \tag{6}$$

Where $u \in \mathbb{R}^{n-p} = \mathbb{R}^{n_1 - 1} \times \ldots \times \mathbb{R}^{n_p - 1}$ is considered as $u = (u^\alpha)_\alpha$ with $u^\alpha := (u_1^\alpha, \ldots, u_{n_\alpha - 1}^\alpha)$.

For every $u \in \mathbb{R}^{n-p}$ and $x = \phi(u)$, we have

$$(\mathrm{d}_u\phi)B(\mathrm{d}_u\phi)^t = (-1)\, T_x D_x A D_x T_x^t = \pi_A(x)\ , \tag{7}$$

where $\mathrm{d}_u\phi$ denotes the jacobian of ϕ at point u and $B := (-1)EAE^t$ where E is the block diagonal matrix $E := \mathrm{diag}(E_1, \ldots, E_p)$ in which E_α is the $(n_\alpha - 1) \times n_\alpha$ block matrix defined as follows

$$E_\alpha := \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, \quad \alpha = 1, \ldots, p \tag{8}$$

The condition of A being anti-symmetric can be relaxed a bit in the sense that:
If there exist a diagonal matrix D and an anti-symetric matrix A_0 such that $A = A_0 D$, Then the polymtrix replicator $X_{A,\underline{n}}$ is Hamiltonian with respect to Poisson structure π_{A_0} having $\tilde{H} = H \circ D$ as its Hamiltonian.

In a recent work, which will be on arxiv soon, we extended this result using Dirac\big-isotropic structures, i.e. introduced Hamiltonian polymatirx replicators with Dirac\big-isotropic structures as underlying geometry.

$$\text{Dirac}: \quad L = L^{\perp}$$

$$\text{big-isotropic}: \quad L \subset L^{\perp}$$

In the context of Poisson geometry:

Generic Hamiltonian Polymatrix replicators

$$\Downarrow$$

Hamiltonian skeleton character

Let

$$(A_1)_i^\alpha := \sum_{\beta\neq\alpha}(A^{\alpha,\beta}x^\beta)_i + \sum_{k\neq i} a_{ik}^{\alpha,\alpha} x_k^\alpha - \sum_{\beta\neq\alpha}\sum_{k\neq i} x_k^\alpha (A^{\alpha,\beta}x^\beta)_k - \sum_{k,l\neq i} x_k^\alpha a_{k,l}^{\alpha,\alpha} x_l^\alpha \,, \tag{9}$$

$$(A_2)_i^\alpha := a_{i,i}^{\alpha,\alpha} - \sum_{\beta\neq\alpha}(A^{\alpha,\beta}x^\beta)_i - \sum_{k\neq i}(a_{i,k}^{\alpha,\alpha} + a_{k,i}^{\alpha,\alpha})x_k \,, \tag{10}$$

Proposition

Let X_A be a polymatrix replicator defined on $\Gamma_{\underline{n}}$. Then for every face $\sigma_i^\alpha \in F$

$$\nu(X_A, \sigma_i^\alpha) = \begin{cases} 3 & \text{if the polynomials (9) and (10) vanish} \\ 2 & \text{if the polynomial (9) vanishes} \\ 1 & \text{otherwise,} \end{cases}$$

By generic, I mean Hamiltonian polymatrix replicators where

$$\nu(X_A, \sigma_i^\alpha) = 1 \quad \text{for all faces}$$

Poisson Poincare maps I

Let (M, π) be a Poisson manifold and denote by $\mathcal{F}$ its symplectic foliation. A submanifold $N \subset M$ is called a *Poisson-Dirac submanifold* if N is a poisson manifold and

i) The symplectic foliation of N is $\mathcal{F}_N = N \cap \mathcal{F}$.

ii) For every leaf $L \in \mathcal{F},\ L \cap N$ is a symplectic submanifold of L.

Poisson Poincare maps II

Also, the rank of the Poisson-Dirac submanifold N at point $x \in N$ is defined to be the number

$$(\mathrm{rank}\, N)_x = \dim(T_x N^\circ \cap \mathrm{Ker}\pi_x^\sharp),$$

where $\pi^\sharp : T^*M \to TM$ is the vector bundle map defined by $\xi \to \pi(\xi, .)$ and

$$T_x N^\circ := \{\xi \in T_x^* M |\ \xi(v) = 0\ \forall v \in T_x N\}$$

is the annihilator of $T_x N$ in $T_x^* M$.

Since a Poisson manifold is determined completely by its symplectic foliation, every Poisson-Dirac submanifold N of a Poisson manifold M has a unique Poisson structure that is been induced by the Poisson structure of M. In the language of Poisson brackets, one can compute the induced Poisson structure π_N as follows

$$\pi_N(\mathrm{d}f_1, \mathrm{d}f_2) = \{f_1, f_2\}_N = \{\tilde{f}_1, \tilde{f}_2\}|_N, \tag{11}$$

where $\tilde{f}_i$ is an extension of $f_i \in C^\infty(N)$ to M such that $\mathrm{d}\tilde{f}_i|_{\pi^\sharp(TN^\circ)} = 0$ for $i = 1, 2$.

Definition

A submanifold N of (M, π) is called *pointwise Poisson-Dirac* submanifold if and only if for every $x \in N$,

$$T_x N \cap \pi^\sharp(T_x N^\circ) = \{0\}.$$

Lemma

A pointwise Poisson-Dirac submanifold N is Poisson-Dirac if the the rank of N is constant all over N.

Consider a Hamiltonian H on the m-dimensional Poisson manifold (M, π) and fix a point $x_0 \in M$ such that $X_H(x) \neq 0$. Consider a $(m-2)$ dimensional transversal section to X_H at point x_0

$$\Sigma \subset H^{-1}(e_1) \cap U,$$

where $H(x_0) = e_1$.

Let dG be the 1−form in which $\{d_x H, d_x G\}$ is a basis for $T_x\Sigma^\circ$ (shrink U if necessary). Then

$$T_\Sigma M = T\Sigma \oplus \mathbb{R}X_H \oplus \mathbb{R}X_G. \qquad (12)$$

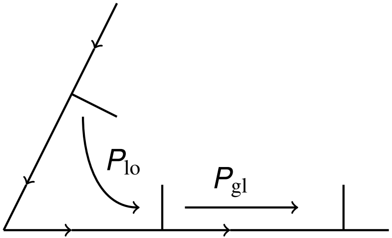
Lemma

Every transversal section Σ defined as above is a Poisson-Dirac submanifold of M.

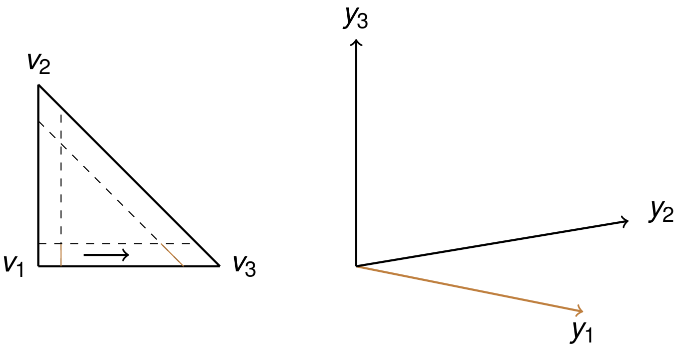
For a fixed time t_0, let $x_1 = \phi_H(t_0, x_0)$, where ϕ_H is the flow of the Hamiltonian vector field X_H, and Σ_0, Σ_1 are transversal sections at x_0 and x_1, respectively.
As usual, a Poincaré map $P = \phi_H(\tau(x), x)$ can be defined from an appropriate neighbourhood of x_0 in Σ_0 to a neighborhood of x_1 in Σ_1.

Proposition

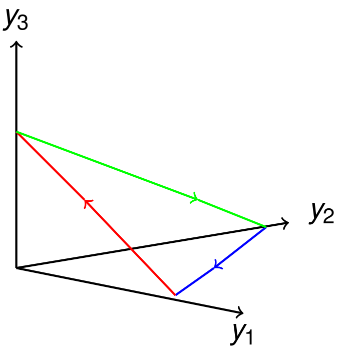
The Poincaré map $P : (\Sigma_0, \pi_{\Sigma_0}) \to (\Sigma_1, \pi_{\Sigma_1})$, where π_{Σ_i} $i = 0, 1$ are the induced Poisson structures defined at (11), is a Poisson map.



The payoff matrix A being anti-symmetric, the map P_{lo} can be defined from a co-dimension 2 section Σ_0 to a codimension 2 section Σ_1 and it is a Poisson map.



If all the faces of $\Gamma_{\underline{n}}$ has degree one, the subspace of the gray line associated to Σ_0 inherits a Poisson structure.



And the asymptotic Poincare map is a time one map of a Hamiltonian flow i.e. the Hamiltonian character of the dynamics is inherited by the asymptotic Poincare map.

This simplifes life by dropping the dimension to the dimension of the symplectic leaves. We found examples of Hamiltonian vector field where the study of the asymptotic flow reveals existence of chaotic behavior.

Let $p = 1$ and $\underline{n} = (5)$ and

$$A = \begin{bmatrix} 0 & -2 & 2 & -2 & 2 \\ 2 & 0 & -2 & 0 & 0 \\ -2 & 2 & 0 & -3 & 0 \\ 2 & 0 & 3 & 0 & -2 \\ -2 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

The point

$$q = \left(\frac{1}{8}, \frac{5}{16}, \frac{1}{8}, \frac{1}{8}, \frac{5}{16} \right),$$

is an equilibrium point of the replicator equation $X_{(A,\underline{n})}$.

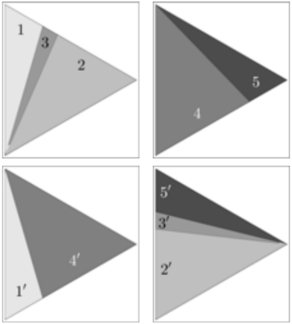


Figure: The pictures above represent the sets $\Delta_{\xi_1}, \ldots, \Delta_{\xi_5}$, where $\xi_1, ..\xi_5$ are χ-pathes. The pictures below represent the π_ξ iterates of $\Delta_{\xi_1}, \ldots, \Delta_{\xi_5}$, labelled from $1'$ to $5'$.

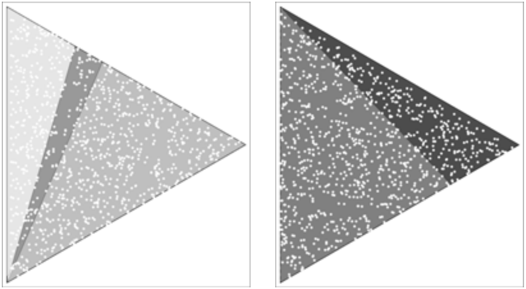


Figure: One orbit with 2000 iterates.

Let $\underline{n} = (2, 2, 2)$ and

$$A = \begin{bmatrix} 0 & -5/2 & 0 & 9/8 & 0 & 2 \\ 5/2 & 0 & 0 & -9/8 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ -5/4 & 5/4 & 0 & 0 & 0 & 0 \\ 0 & 5/2 & -9/8 & 0 & 0 & -1 \\ -5/2 & -5/4 & 9/4 & 0 & 1 & 0 \end{bmatrix}.$$

$A = A_0\, D$, where $D = \mathrm{diag}\left(\frac{5}{2}, \frac{5}{2}, \frac{9}{4}, \frac{9}{4}, 2, 2\right)$ and

$$A_0 = \begin{bmatrix} 0 & -1 & 0 & \frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

$q = (7/10, 3/10, 5/9, 4/9, 1/2, 1/2)$ is a formal equilibrium for A. So $(\underline{n}, A)$ is conservative. The point q is inside $\Gamma_{\underline{n}}$ so it is an actual equilibrium point.

The phase space of the associated replicator system is the cube

$$\Gamma_{(2,2,2)} = \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3 \ .$$

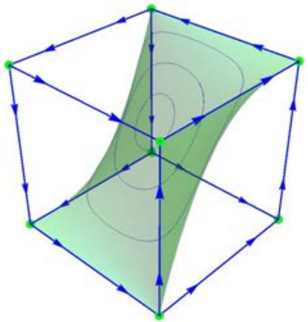


Figure: Phase portraits of 3D polymatrix replicator 1

An ohter example where $\underline{n} = (3, 2)$ so

$$\Gamma_{(3,2)} = \Delta^2 \times \Delta^1 \equiv \{ (x, y, z) \, : \, 0 \leq x, y, z \leq 1, \; x + y \leq 1 \}.$$

We considered a system with a formal equilibrium outside $\Gamma_{\underline{n}}$.

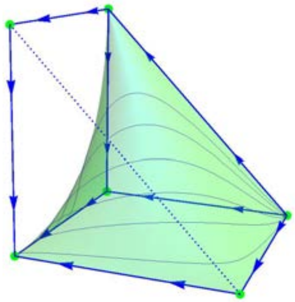


Figure: Phase portraits of 3D polymatrix replicator 2

Gracias!