Multisymplectic approach to Symmetries in Field Theories

Jordi Gaset

Universitat Politècnica de Catalunya

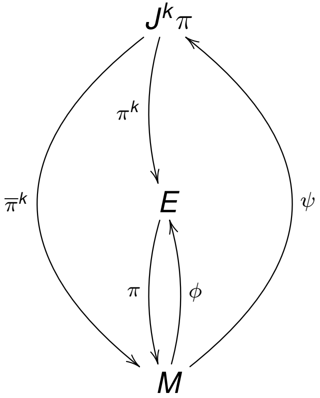
VI Iberoamerican meeting,
Geometry, Mechanics and Control
August, 2018

Table of contents

1. Multisymplectic field theories
2. Symmetries, Gauge freedom and Conserved Quantities
3. Examples: Metric-Affine model of General Relativity

Jet Bundles

Fibre bundle $\pi : E \to M$.
$\dim M = m$, $\dim E = n + m$.
k-jet manifold is $J^k\pi$.
Natural coordinates of $J^k\pi$:
$(x^i, u^\alpha, u^\alpha_{i_1,\dots,i_r})$,
$1 \le i_1 \le \cdots \le i_r \le m$,
$1 \le r \le k$, $1 \le \alpha \le n$,



Multivector Fields

Curve $\gamma \longrightarrow$ 1-distribution $\longrightarrow$ vector field

Multivector Fields

Curve	γ	$\longrightarrow$	1-distribution	$\longrightarrow$	vector field
Field	ψ	$\longrightarrow$	*m*-distribution		

Multivector Fields

Curve	γ	$\longrightarrow$	1-distribution	$\longrightarrow$	vector field
Field	ψ	$\longrightarrow$	*m*-distribution	$\longrightarrow$	multivector field

Multivector Fields

Curve	γ	$\longrightarrow$	1-distribution	$\longrightarrow$	vector field
Field	ψ	$\longrightarrow$	m-distribution	$\longrightarrow$	multivector field

Multivector fields

An m-multivector field in $J^k\pi$ is a skew-symmetric contravariant tensor of order m in $J^k\pi$.

- Locally decomposable if, for every $p \in J^k\pi$, there is an open neighbourhood $U_p \subset J^k\pi$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$. They are locally associated with m-dimensional distributions $D \subset \mathrm{T}J^k\pi$.
- Integrable if its associated distribution is integrable.
- Holonomic if it is integrable and its integral sections are holonomic sections of $\bar{\pi}^k$.

Multivector Fields

Curve	γ	$\longrightarrow$	1-distribution	$\longrightarrow$	vector field
Field	ψ	$\longrightarrow$	m-distribution	$\longrightarrow$	multivector field

Multivector fields

An m-multivector field in $J^k\pi$ is a skew-symmetric contravariant tensor of order m in $J^k\pi$.

- Locally decomposable if, for every $p \in J^k\pi$, there is an open neighbourhood $U_p \subset J^k\pi$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$. They are locally associated with m-dimensional distributions $D \subset \mathrm{T}J^k\pi$.
- Integrable if its associated distribution is integrable.
- Holonomic if it is integrable and its integral sections are holonomic sections of $\bar{\pi}^k$.

$$\mathbf{X} = \bigwedge_{\nu=1}^{m} X_\nu = \bigwedge_{\nu=1}^{m} \left(\frac{\partial}{\partial x^\nu} + Y_\nu \right)$$

Multisymplectic formalism

(pre)-multisymplectic forms

A multisymplectic form is a closed 1-non-degenerate (m+1)-form $\Omega \in \mathit{\Omega}^{m+1}(J)$.
It is pre-multisymplectic if it is closed but 1-degenerate.

Multisymplectic formalism

(pre)-multisymplectic forms

A multisymplectic form is a closed 1-non-degenerate (m+1)-form $\Omega \in \Omega^{m+1}(J)$.
It is pre-multisymplectic if it is closed but 1-degenerate.

Lagrangian: $\mathcal{L} \in C^\infty(J^s\pi)$, then $J^k\pi$, $k = 2s - 1$
Poincaré-Cartan $m + 1$ form: $\Omega_\mathcal{L} = -d\Theta_\mathcal{L} \in \Omega^{m+1}(J^k\pi)$

Multisymplectic formalism

(pre)-multisymplectic forms

A multisymplectic form is a closed 1-non-degenerate (m+1)-form $\Omega \in \Omega^{m+1}(J)$.
It is pre-multisymplectic if it is closed but 1-degenerate.

Lagrangian: $\mathcal{L} \in C^\infty(J^s\pi)$, then $J^k\pi$, $k = 2s - 1$
Poincaré-Cartan $m+1$ form: $\Omega_\mathcal{L} = -d\Theta_\mathcal{L} \in \Omega^{m+1}(J^k\pi)$

Lagrangian multisymplectic problem

- Find holonomic sections $\psi : M \to J^k$ such that

$$\psi^* \, i(Y)\Omega_r = 0 \, , \quad \forall Y \in \mathfrak{X}(J^k) \, .$$

- Finding a holonomic multivector field $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ such that:

$$i(\mathbf{X})\Omega_\mathcal{L} = 0.$$

Multisymplectic formalism

(pre)-multisymplectic forms

A multisymplectic form is a closed 1-non-degenerate (m+1)-form $\Omega \in \Omega^{m+1}(J)$.
It is pre-multisymplectic if it is closed but 1-degenerate.

Lagrangian: $\mathcal{L} \in C^\infty(J^s\pi)$, then $J^k\pi$, $k = 2s - 1$
Poincaré-Cartan $m+1$ form: $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^k\pi)$

Lagrangian multisymplectic problem

- Find holonomic sections $\psi : M \to J^k$ such that

$$\psi^* \, i(Y)\Omega_r = 0\,, \quad \forall Y \in \mathfrak{X}(J^k)\,.$$

- Finding a holonomic multivector field $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ such that:

$$i(\mathbf{X})\Omega_{\mathcal{L}} = 0.$$

$Sol_f \equiv$ sections solutions of the multisymplectic problem, but not necessary holonomic.

Symmetries

Symmetries

A symmetry of a Lagrangian system $(J^k\pi, \Omega_{\mathcal{L}})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ such that $\Phi_*(Sol_f) \subset Sol_f$.
An infinitesimal symmetry of a Lagrangian system $(J^k\pi, \Omega_{\mathcal{L}})$ is a vector field $Y \in \mathfrak{X}(J^k\pi)$ whose local flows are local symmetries.

Symmetries

Symmetries

A symmetry of a Lagrangian system $(J^k\pi, \Omega_{\mathcal{L}})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ such that $\Phi_*(Sol_f) \subset Sol_f$.
An infinitesimal symmetry of a Lagrangian system $(J^k\pi, \Omega_{\mathcal{L}})$ is a vector field $Y \in \mathfrak{X}(J^k\pi)$ whose local flows are local symmetries.

Cartan	$\Phi^*\Omega_{\mathcal{L}} = \Omega_{\mathcal{L}}$	$\mathrm{L}(Y)\Omega_{\mathcal{L}} = 0$
Exact Cartan	$\Phi^*\Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$	$\mathrm{L}(Y)\Theta_{\mathcal{L}} = 0$
Lagrangian	$\Phi^*\mathcal{L} = \mathcal{L}$	$\mathrm{L}(Y)\mathcal{L} = 0$
Geometric Gauge		$i(Y)\Omega_{\mathcal{L}} = 0$ and π-vertical.

Symmetries

A symmetry of a Lagrangian system $(J^k\pi, \Omega_{\mathcal{L}})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ such that $\Phi_*(Sol_f) \subset Sol_f$.
An infinitesimal symmetry of a Lagrangian system $(J^k\pi, \Omega_{\mathcal{L}})$ is a vector field $Y \in \mathfrak{X}(J^k\pi)$ whose local flows are local symmetries.

Cartan	$\Phi^*\Omega_{\mathcal{L}} = \Omega_{\mathcal{L}}$	$\mathrm{L}(Y)\Omega_{\mathcal{L}} = 0$
Exact Cartan	$\Phi^*\Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$	$\mathrm{L}(Y)\Theta_{\mathcal{L}} = 0$
Lagrangian	$\Phi^*\mathcal{L} = \mathcal{L}$	$\mathrm{L}(Y)\mathcal{L} = 0$
Geometric Gauge		$i(Y)\Omega_{\mathcal{L}} = 0$ and π-vertical.

Geometric Gauge Freedom

- We can define the multisymplectic quotient system because $\Omega_{\mathcal{L}}$ is projectable to the quotient.

Geometric Gauge Freedom

- We can define the multisymplectic quotient system because $\Omega_{\mathcal{L}}$ is projectable to the quotient.
- We can define gauge related sections and distributions.

Geometric Gauge Freedom

- We can define the multisymplectic quotient system because $\Omega_{\mathcal{L}}$ is projectable to the quotient.
- We can define gauge related sections and distributions.
- Multiple distributions which are not gauge related:

Lagrangian problem for multivector fields

$m = 1$ Multiple solutions $\Leftrightarrow$ geometric gauge freedom.
$m > 1$ Multiple solutions $\Leftarrow$ geometric gauge freedom.

Conserved Quantities

Conserved Quantities

A conserved quantity is a form $\xi \in \Omega^{m-1}(J^k\pi)$ such that, for every $\psi \in Sol_f$

$$d\psi^*\xi = 0.$$

Conserved Quantities

Conserved Quantities

A conserved quantity is a form $\xi \in \Omega^{m-1}(J^k\pi)$ such that, for every $\psi \in Sol_f$

$$\mathrm{d}\psi^*\xi = 0.$$

Noether's Theorem

If $Y \in \mathfrak{X}(J^k\pi)$ is an infinitesimal Cartan symmetry, with $i(Y)\Omega_{\mathcal{L}} = \mathrm{d}\xi_Y$ then ξ_Y is a conserved quantity.

Conserved Quantities

Conserved Quantities

A conserved quantity is a form $\xi \in \Omega^{m-1}(J^k\pi)$ such that, for every $\psi \in Sol_f$

$$\mathrm{d}\psi^*\xi = 0.$$

Noether's Theorem

If $Y \in \mathfrak{X}(J^k\pi)$ is an infinitesimal Cartan symmetry, with $i(Y)\Omega_{\mathcal{L}} = \mathrm{d}\xi_Y$ then ξ_Y is a conserved quantity.

Currents conservation

For ξ conserved quantity, $\psi \in Sol_f$, and $U \subset M$ a bounded domain, then:

$$\int_{\partial U} \psi^*\xi = 0$$

The form $\psi^*\xi$ is the current associated to ξ (by ψ).

Metric-Affine Lagrangian

M 4-diemnsional connex manifold.

Fiber bundle $\pi : E = \Sigma \times_M C(LM) \to M$,

Σ is the manifold of Lorentzian metrics,

$C(LM)$ manifold of lineal connections of $\mathrm{T}M$.

The induced coordinates in $J^1\pi$ are $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, \Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_{\beta\gamma,\mu})$.

Metric-Affine Lagrangian

M 4-diemnsional connex manifold.

Fiber bundle $\pi : E = \Sigma \times_M C(LM) \to M$,

Σ is the manifold of Lorentzian metrics,

$C(LM)$ manifold of lineal connections of $\mathrm{T}M$.

The induced coordinates in $J^1\pi$ are $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, \Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_{\beta\gamma,\mu})$.

The Hilbert-Einsten Lagrangian function without matter

$$L_{MA} = \sqrt{|\det(g)|}\, g^{\alpha\beta} R_{\alpha\beta} = \sqrt{|\det(g)|}\, g^{\alpha\beta} \left(\Gamma^\gamma_{\beta\alpha,\gamma} - \Gamma^\gamma_{\gamma\alpha,\beta} + \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma} - \Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha} \right) \in \Omega^5(J^1\pi)$$

M 4-diemnsional connex manifold.

Fiber bundle $\pi : E = \Sigma \times_M C(LM) \to M$,

Σ is the manifold of Lorentzian metrics,

$C(LM)$ manifold of lineal connections of $\mathrm{T}M$.

The induced coordinates in $J^1\pi$ are $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, \Gamma^\alpha_{\beta\gamma}, \Gamma^\alpha_{\beta\gamma,\mu})$.

The Hilbert-Einsten Lagrangian function without matter

$$L_{MA} = \sqrt{|\det(g)|}\, g^{\alpha\beta} R_{\alpha\beta} = \sqrt{|\det(g)|}\, g^{\alpha\beta} \left(\Gamma^\gamma_{\beta\alpha,\gamma} - \Gamma^\gamma_{\gamma\alpha,\beta} + \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma} - \Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha} \right) \in \Omega^5(J^1\pi)$$

$$\Omega_{L_{\mathrm{MA}}} = \mathrm{d}\left(\frac{\partial L_{\mathrm{MA}}}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \Gamma^\alpha_{\beta\gamma,\mu} - L_{\mathrm{MA}} \right) \wedge \mathrm{d}^4 x - \mathrm{d}\frac{\partial L_{\mathrm{MA}}}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \wedge \mathrm{d}\Gamma^\alpha_{\beta\gamma} \wedge \mathrm{d}^3 x_\mu \ .$$

Metric-Affine Lagrangian

The geometric gauge vector fields are,

$$Y = C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + C_{\beta,\mu} \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}}, \quad \forall C_\beta, C_{\beta,\mu} \in C^\infty(J^1\pi),$$

Metric-Affine Lagrangian

The geometric gauge vector fields are,

$$Y = C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + C_{\beta,\mu} \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}}, \quad \forall C_\beta, C_{\beta,\mu} \in C^\infty(J^1\pi),$$

Lifting of diffeomorphisms

The lifting of $\varphi : N \to M$ is

$$\begin{array}{rcl} \phi : \Sigma \times_M C(LM) & \to & \Sigma \times_M C(LM) \\ (x, g_x, \Gamma_x) & \mapsto & \phi(x, g_x, \Gamma_x) = (x, \varphi_* g_x, \varphi_* \Gamma_x) \end{array}$$

$$\begin{array}{ccccc} J^1\pi & \xrightarrow{\pi^1} & \Sigma \times_M C(LM) & \xrightarrow{\pi} & M \\ \downarrow \Phi & & \downarrow \phi & & \downarrow \varphi \\ J^1\pi & \xrightarrow[\pi^1]{} & \Sigma \times_M C(LM) & \xrightarrow[\pi]{} & M \end{array}$$

Metric-Affine Lagrangian

The geometric gauge vector fields are,

$$Y = C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + C_{\beta,\mu} \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}}, \quad \forall C_\beta, C_{\beta,\mu} \in C^\infty(J^1\pi),$$

Lifting of diffeomorphisms

The lifting of $\varphi : N \to M$ is

$$\begin{array}{rcl} \phi : \Sigma \times_M C(LM) & \to & \Sigma \times_M C(LM) \\ (x, g_x, \Gamma_x) & \mapsto & \phi(x, g_x, \Gamma_x) = (x, \varphi_* g_x, \varphi_* \Gamma_x) \end{array}$$

$$\begin{array}{ccccc} J^1\pi & \xrightarrow{\pi^1} & \Sigma \times_M C(LM) & \xrightarrow{\pi} & M \\ \Big\downarrow \Phi & & \Big\downarrow \phi & & \Big\downarrow \varphi \\ J^1\pi & \xrightarrow[\pi^1]{} & \Sigma \times_M C(LM) & \xrightarrow[\pi]{} & M \end{array}$$

The canonical lifting to $J^1\pi$ of liftings of diffeomorphisms are Lagrangian symmetries.

Metric-Affine Lagrangian

$$\mathbf{X} = \bigwedge_{\nu=0}^{3} \left(\frac{\partial}{\partial x^\nu} + \sum_{\rho\leq\sigma} \left(g_{\rho\sigma,\nu} \frac{\partial}{\partial g_{\rho\sigma}} + F_{\rho\sigma\mu,\nu} \frac{\partial}{\partial g_{\rho\sigma,\mu}} \right) + \Gamma^{\alpha}_{\beta\gamma,\nu} \frac{\partial}{\partial \Gamma^{\alpha}_{\beta\gamma}} + F^{\alpha}_{\beta\gamma\mu,\nu} \frac{\partial}{\partial \Gamma^{\alpha}_{\beta\gamma,\mu}} \right) \, ,$$

where,

$$\begin{aligned} F_{\rho\sigma\mu,\nu} &= D_\nu \left(g_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho} + g_{\rho\lambda}\Gamma^{\lambda}_{\mu\sigma} + \frac{2}{3} g_{\rho\sigma} T^{\lambda}_{\lambda\mu} \right) \, , \\ F^{\alpha}_{\beta\gamma\mu,\nu} &= \Gamma^{\lambda}_{\mu\gamma,\nu}\Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\lambda}_{\mu\gamma}\Gamma^{\alpha}_{\beta\lambda,\nu} + C_{\beta\mu\nu}\delta^{\alpha}_{\gamma} + K^{\alpha}_{\beta\gamma,\mu\nu} \end{aligned}$$

For any $C_{\beta\mu\nu} \in C^\infty(J^1\pi)$ and $K^{\alpha}_{\beta\gamma,\mu\nu} \in C^\infty(J^1\pi)$ such that,

$$\begin{aligned} K^{\lambda}_{\lambda\gamma,\mu\nu} &= 0 \quad , \quad K^{\lambda}_{\beta\gamma,\lambda\nu} + K^{\lambda}_{\gamma\beta,\lambda\nu} = 0 \, ; \\ K^{\alpha}_{[\beta\gamma],\mu\nu} &= -\frac{1}{3}\delta^{\alpha}_{[\beta} K^{\lambda}_{\gamma]\lambda,\mu\nu} - \Gamma^{\lambda}_{\mu[\gamma,\nu}\Gamma^{\alpha}_{\beta]\lambda} - \Gamma^{\lambda}_{\mu[\gamma}\Gamma^{\alpha}_{\beta]\lambda,\nu} \\ &+ \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\gamma],\nu}\Gamma^{\rho}_{\rho\lambda} + \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\gamma]}\Gamma^{\rho}_{\rho\lambda,\nu} - \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\rho,\nu}\Gamma^{\rho}_{\gamma]\lambda} - \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\gamma]\lambda,\nu} \, ; \\ g_{\alpha\lambda}K^{\lambda}_{[\nu\beta\mu],\xi} &+ g_{\beta\lambda}K^{\lambda}_{[\nu\alpha\mu],\xi} = -2g_{\alpha\beta,\xi}T^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\sigma\lambda} - 2g_{\alpha\beta}T^{\lambda}_{\mu\nu,\xi}\Gamma^{\sigma}_{\sigma\lambda} - 2g_{\alpha\beta}T^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\sigma\lambda,\xi} \\ &\quad -g_{\alpha\lambda,\xi}K^{\lambda}_{[\nu\beta\mu]} - g_{\beta\lambda,\xi}K^{\lambda}_{[\nu\alpha\mu]} \, . \end{aligned}$$

Bibliography

- N. Dadhich, J.M. Pons, “On the equivalence of the Einstein–Hilbert and the Einstein–Palatini formulations of general relativity for an arbitrary connection”, *Gen. Rel. Grav.* **44**(9) (2012) 2337-2352. (doi: 10.1007/s10714-012-1393-9).
- D. Krupka and O. Stepankova, “On the Hamilton form in second order calculus of variations”, in: *Proc. Internat. Meeting ”Geometry and Physics”*, Florence, October 1982, Pitagora, Bologna, 1983, 85-101.
- E. Rosado María, J. Muñoz, “Integrability of second-order Lagrangians admitting a first-order Hamiltonian formalism”, *Differential Geometry and its Applications*, Volume 35, Supplement, September 2014, Pages 164–177.
- J. Gaset, N. Román-Roy, “Multisymplectic unified formalism for Einstein-Hilbert Gravity”. arXiv:1705.00569 (2017).
- J. Gaset, P.D. Prieto-Martínez, N. Román-Roy, “Variational principles and symmetries on fibered multisymplectic manifolds”. *Communications in Mathematics*, Volume 24, Issue 2, Pages 137-152; 2016.