Perception of images by the visual cortex: geometry in neuroscience

Pascal Chossat*

VI Iberoamerican meeting “Geometry, Mechanics and Control”

In the honor of James Montaldi

CIMAT, Guanajuato

(*)
The visual cortex is a multiscale complex system
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How does the brain process the input signal from retina to give a global and as much as possible coherent representation of the outer world (the ”Gestalt”)? Indeed, informations from retina to visual cortex are essentially local...
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Visual hallucinations
Outline of the talk

1. Functional architecture and geometry of the primary visual cortex.
   - Detection of contours: the primary visual cortex as a contact structure
   - Application to the occurrence of geometric hallucinations under non-visual stimulation like drugs...
   - Introducing scale as a new feature: the primary visual cortex as a symplectic structure (details in Sarti, Citti & Petitot 2008).

2. Extending 1 to contrast & selectivity: the "hyperbolic" model.
   - A recent attempt to incorporate more features to the ring model by replacing "orientation" with "structure tensor" (Gregory Faye's thesis, Inria 2013) → pattern formation problem in the hyperbolic plane.
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- Neurons are **active units**, which emit **spike trains** along axon when input exceeds a threshold.
- Synapses are either **excitatory** (green) or **inhibitory** (red).

- In a specific brain area there are millions of neurons.
- It is relevant to consider space and time averages of the activity → continuous time evolution of **neural fields** (as measured in ECG, FMRI). Then ”neuron” means in fact ”population of neurons”.
- Synaptic plasticity allows reconfiguration of circuitry at various time scales (long-term and short-term learning, adaptation..).
Global structure of the visual cortex in primates

Signal generated on retina is transmitted to the primary visual area \( V_1 \) after filtering (smoothing) in the LGN. Then forwarded to other areas \( V_2, V_3, \ldots \).

Each neuron in \( V_1 \) responds to a local receptive field in the visual field \( VF \) where it detects orientation, contrast, spatial frequency, ocular dominance...

Typical receptive profile \( \phi \) of a neuron in \( V_1 \):

It filters the signal \( I(x, y) \) at a spatial scale \( \sigma \) with a local orientation:

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I_{\phi} = I \ast \phi
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(for example \( \phi = \partial_x^2 G \), \( G(x, y) \) Gauss function with standard deviation \( \sigma \)).
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Typical receptive profile $\varphi$ of a neuron in $V1$:
It filters the signal $I(x, y)$ at a spatial scale $\sigma$ with a local orientation: $I_\varphi = I \ast \varphi$ (for example $\varphi = \partial_x^2 G$, $G(x, y)$ Gauss function with standard deviation $\sigma$).
Small patches in the visual field VF are mapped to small patches in V1 according to a roughly $\log(z)$ law ($z \in \mathbb{C} \simeq VF$).

This retinotopic map $VF \rightarrow V1$ is an approximately conformal map. The fovea is mapped to a large domain in V1.

Retinotopic map for a macaque
Columnar structure of V1 (Hubel & Wiesel 1960's)

1. They got the Nobel prize in physiology (1981) for this discovery.

2. \( V_1 \) is composed of six "horizontal" layers. Experiments show:
   - Neurons in a vertical column detect the same orientation, except at singular columns called pinwheels where all orientations are present.
   - Neurons in adjacent columns detect different orientations by steps of \( \sim 10^\circ \).

3. The patch of adjacent columns surrounding a pinwheel defines a hypercolumn (\( \sim 0.6 \text{ mm}^2 \)), in which neurons respond to the same location in retina but to different orientations.

4. Other features are engrafted in hypercolumns: contrast, spatial frequency, ocular dominance, that could be accounted for as well.
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[Diagram of columnar structure]
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Hypercolumnar crystalline structure of V1

(i) Diametrically opposite rays correspond to orientations differing by $\pi/2$.

(ii) Pinwheels form a crystal lattice on V1.

(iii) Iso-orientation lines define a field of orientations, of which pinwheels are singular points.

(iv) Tempting to idealize the hypercolumnar structure by a fiber bundle structure $R \times \mathbb{P}^1 \cong J_1 R$: $R$ = retinal field (base plane) and fiber = set of orientations $\cong$ projective line.

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Let \( \gamma \) be a curve in \( \mathbb{R} \simeq \mathbb{R}^2 \) with tangent angle \( \theta \) at \((x, y)\). \( \gamma \) lifts to the curve \( \Gamma = \{(x, y, \theta)\} \) in \( J^1\mathbb{R} \simeq \mathbb{R} \times S^1 \) (\( \theta \in [0, \pi] \)). This allows to replace the evaluation of \( dy/dx \) at each \((x, y)\) in \( \gamma \) by the selection of a point in the fiber bundle: much more efficient!

\( \Gamma \) is the lift of a curve in \( \mathbb{R} \) if any tangent vector to \( \Gamma \) belongs to the kernel of the differential 1-form \( \omega = \cos(\theta) dy - \sin(\theta) dx \).

\( \ker \omega \) defines a distribution of planes (called horizontal) in \( T (\mathbb{R} \times S^1) \), spanned at each point \((x, y, \theta)\) by \( \cos(\theta) \partial_x + \sin(\theta) \partial_y \) and \( \partial_\theta \).

This distribution of planes defines a contact structure: \( \omega \wedge d\omega > 0 \Rightarrow \) it is not integrable in \( \mathbb{R} \times S^1 \) (Froebenius theorem). It is devoted to path (contour) integration.

It is tempting to say that pinwheels and their iso-orientation rays represent a discrete neural implementation of this contact structure. But how does the brain proceed to compare orientations at remote points in \( \mathbb{R}^2 \)?
Let $\gamma$ be a curve in $R \simeq \mathbb{R}^2$ with tangent angle $\theta$ at $(x, y)$. 
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The fiber bundle $V1$ as an efficient detector of contours

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Long-range horizontal connections in V1

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(i) Neurons with long-range axons connect preferentially to neurons with the same orientation in other hypercolumns.
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In contrast, note that within the hypercolumn the field of connections looks quite isotropic: it equally reaches all orientations.
$SE(2) = \mathbb{R}^2 \ltimes SO(2)$ is the Lie group of rigid displacements in the plane, with product $(t, r_\phi) \cdot (t', r_{\phi'}) = (t + r_\phi t', r_{\phi + \phi'})$. 

\footnotesize{Pascal Chossat* 

Citti & Sarti 2006, Petitot 2003 → nice formulation using the subriemannian metric induced by the contact structure.}
V1 as the Lie group of displacements in the plane

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- $SE(2)$ is the principal bundle associated with $R \times \mathbb{P}^1$: $\mathbb{R}^2 \simeq SE(2)/SO(2)$. 

Remark. This is different from Mumford’s Elastica formulation, which sits in the base plane $\mathbb{R}$ and minimizes an energy $\gamma^\alpha \kappa^\beta$ where $\kappa$ is the curvature.
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- Then $\omega$ is invariant under the action of $SE(2)$ on $T(\mathbb{R}^2 \times SO(2))$: any two horizontal planes in $\mathbb{R}^2 \times SO(2)$ can be equivariantly identified by a suitable displacement.

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- Application to illusory contours: problem of minimizing length in $J^1_{\mathbb{R}}$ or $SE(2)$ under constraint $\ker \omega = 0$.

  Citti & Sarti 2006, Petitot 2003 —> nice formulation using the subriemannian metric induced by the contact structure.

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Geometric hallucinations as a spontaneous activity in V1

Drugs diffusing homogeneously in the brain can induce visual hallucinatory patterns. Examples under marijuana or LSD:

(from Bressloff et al 2001)
Bressloff, Cowan & Golubitsky theory of hallucinations (2001)

Wilson-Cowan equation for the averaged action potential of neural field:

\[
\frac{da(x, \theta, t)}{dt} = -a(x, \theta, t) + \int_{\mathbb{R}^2} \int_0^{2\pi} w(x, \theta; x', \theta') S_{\mu}(a(x', \theta', t)) \, d\theta' \, dx' + I_{\text{ext}} S_{\mu} = \text{sigmoid function, } S_{\mu}(0) = 0, S_{\mu}'(0) = \mu (\text{bifurcation parameter}).
\]

\[
I_{\text{ext}} = 0 \quad (\text{no external input}), \quad w = \text{synaptic strength between neurons.}
\]

The contact structure of \( V_1 \) must be encoded in the function \( w \):

\[
w(x, \theta; x', \theta') = w_{\text{loc}}(\theta, \theta') + w_{\text{lat}}(x, \theta; x', \theta')
\]

where \( w_{\text{loc}} \) is \( S_1 \)-invariant (mod \( \pi \)) and \( w_{\text{lat}} \) is \( E(2) \times S_1 \)-invariant with the orientation and coaxiality constraints of long range neural connections.

Bifurcation analysis `a la Turing`

1. Basic state \( a = 0 \) marginally stable at critical \( \mu = \mu_c \).
2. Look for spatially periodic solutions in \( \mathbb{R}^2 \) (invariant patterns on a rectangular, square or hexagonal lattice \( \Gamma \)).
3. Classical equivariant bifurcation analysis on the torus \( \mathbb{R}^2 / \Gamma \times S_1 \).
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Examples of geometric hallucination

In $\mathbb{R}^2$ plane periodic patterns with or without contours. Applying inverse retinotopic map → image in the visual field.

Hexagonal pattern with no contour

Square pattern with contours
The symplectic structure of V1

The contact form $\omega$ canonically induces a symplectic structure by noting that $r\omega$ induces the same contact form for all $r \in \mathbb{R}^*$ and $(x, y, \theta, r) \in T^* \mathbb{R}$. If we set $\tilde{\omega} = r\omega$, then $d\tilde{\omega}$ is a symplectic form on $T^* \mathbb{R}$ (see Arnold’s Math Methods of Classical Mechanics).
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This structure can explain how $V_1$ spontaneously extracts a medial axis from a contoured figure, a problem considered as fundamental by René Thom, David Mumford, ... : analogue to Huygens principle in optics.
The structure tensor model of V1

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For the image intensity $I(x, y)$ we set $I_\sigma = I \ast g_{\sigma_1}$.
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For the image intensity $I(x,y)$ we set $I_{\sigma_1} = I * g_{\sigma_1}$.

The structure tensor at the point $(x, y)$ is the matrix

$$T(x, y) = g_{\sigma_2} * \left( \nabla I_{\sigma_1} \nabla^T I_{\sigma_1} \right)$$

$\sigma_2$ defines the scale (characteristic size) of the texture to be represented.
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The structure tensor at the point \((x, y)\) is the matrix

\[
\mathcal{T}(x, y) = g_{\sigma_2} * \left( \nabla I_{\sigma_1} \nabla^T I_{\sigma_1} \right)
\]

\( \sigma_2 \) defines the scale (characteristic size) of the texture to be represented.

\( \mathcal{T} \) is a symmetric positive definite matrix: \( \{\mathcal{T}\} \simeq \text{SPD}(2) \).

We shall assume that structure tensors are encoded in the hypercolumns of \( V1 \), so that \( V1 \simeq \text{fiber bundle} \mathbb{R}^2 \times \text{SPD}(2) \).

How does this improve the orientation model, and is it a natural assumption?
Basic properties of the structure tensor

The structure tensor $T$ has two real eigenvalues: $\lambda_1 \geq \lambda_2 > 0$ with eigenvectors $e_1 \perp e_2$.

Elementary algebra shows:

$$T = (\lambda_1 - \lambda_2) e_1 e_1^T + \lambda_2 I_2.$$

- $\lambda_1 \approx \lambda_2 \Rightarrow$ isotropic image
- $\lambda_1 \gg \lambda_2 \approx 0 \Rightarrow$ straight edge along $e_2$
- $\lambda_1 \geq \lambda_2 \gg 0 \Rightarrow$ corner

The size of $\lambda_j$ defines the contrast along $e_j$ the ellipse $x^T x = 1$.

Let $e_2 = (r \cos \theta, r \sin \theta)$, $\theta$ defines the preferred orientation of the neuron.

Note that $T$ is invariant under $\theta \to \theta + \pi$.

In the limit $\lambda_2 = 0$ one recovers the ring model + the contrast along $e_1$.

There are some biological arguments supporting this model for $V_1$, see Faye’s thesis (but direct experimental confirmation is missing).
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the ellipse \( x^T T x = 1 \)

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Riemannian geometry on the set of structure tensors

\[ \text{SPD}(2) \cong Q^+ = \text{space of positive definite quadratic forms } xT x^T \text{ in } \mathbb{R}^2. \]

It is natural to take the metric on \( \text{SPD}(2) \) such that changes of coordinates in \( Q^+ \) leave distances invariant.
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**The following formulation is equivalent and more convenient for our purpose:**

\[ \mathcal{T} = \Delta \tilde{\mathcal{T}} \text{ where } \det \tilde{\mathcal{T}} = 1. \]

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Now, \( \text{SSPD}(2) \cong \text{Lorentz surface } H^2 \cong \text{Poincaré disc } \mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \} \)

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This provides us with a metric on \( \text{SPD}(2) \), for which the distance is

\[
d(T, T') = \sqrt{2 \log^2 \left( \frac{\Delta}{\Delta'} \right) + \text{artanh}^2 \left( \frac{|z - z'|}{|1 - \bar{z}z'|} \right)}
\]

The isometry group is now \( \mathbb{R}^+_* \times U(1, 1), \) where \( U(1, 1) \) acts on \( \mathbb{D} \) by

\[
\gamma z = \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}, \quad |\alpha|^2 - |\beta|^2 = 1, \text{ and reflection } \kappa z = \bar{z}.
\]
We concentrate on the activity in a single hypercolumn (disconnected from others), hence bifurcation from homogeneous state in $\mathbb{R}^+ \times D$. Then discard the $\mathbb{R}^+ \times D$ component (easy part).

- Problem of Turing-like pattern formation in $D$.

Sketch of the method:

1. $\partial_t a(T, t) = -a(T, t) + R D \cdot w(D(T); T') S\mu(a(T', t)) dT'$ where $S\mu$ is sigmoid s.t. $S\mu(0) = 0$, $S'\mu(0) = \mu$.

2. Linear stability analysis: Fourier-Helgason spectral decomp. $\rightarrow$ critical $\mu$.

3. Continuous spectrum $\rightarrow$ look for "periodic" solutions in $D$ to reduce to a problem with discrete spectrum, in order to apply classical equivariant bifurcation methods.

Reference: in the context of parabolic PDEs in $D$, see my paper with G. Faye: Pattern formation for the Swift-Hohenberg equation on the hyperbolic plane, J. Dyn & Diff Equations, Online First (2013).
Spontaneous tuning of structure tensors in V1

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Harmonic and spectral analysis in $\mathbb{D}$

$Iwasa$ Theorem:

$SU(1,1) = KAN$ were $K$, $A$, $N$ are 1-parameter subgroups with orbits rotations hyperbolic boosts parabolic transformations

Harmonic analysis in $\mathbb{D}$ (Fourier-Helgason): based on elementary eigenfunctions $e^{\rho, b}(z) =$ $e^{(i\rho + 1/2)\langle z, b \rangle}$, $\rho \in \mathbb{C}$, where $b \in \partial \mathbb{D}$ and $\langle z, b \rangle$ is a distance built from horocycle based at $b$ and passing by $z$.

It satisfies $-\Delta \mathbb{D} e^{\rho, b} = (\rho^2 + 1/4) e^{\rho, b}$. It allows to build a "Fourier transform" in $\mathbb{D} \rightarrow$ spectral analysis.
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Subgroup of direct isometries (displacements) in $U(1,1)$: pseudo-unitary group $SU(1,1)$ acting in $\mathbb{D}$ by $\gamma z = \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}$, $|\alpha|^2 - |\beta|^2 = 1$. 

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Periodic pattern formation in the Poincaré disc (sketch)

Let $\Gamma \subset SU(1,1)$ be a discrete subgroup which tiles $D$ from a compact fundamental domain $F$. Then $\Gamma$ is spanned by a finite number of hyperbolic boosts. $\Gamma$ is called a co-compact Fuchsian group (or a lattice group). $D/\Gamma \cong$ compact Riemann surface of genus $g \geq 2$ (a torus with $g$ holes).

The $U(1,1)$-invariant equation projects onto $D/\Gamma$ to a $G_\Gamma$-invariant equation where $G_\Gamma$ is the symmetry group of $F_\Gamma$ (seen as a $g$-torus). $G_\Gamma$ is a finite group $\rightarrow$ finite dimensional irreducible representations.

Hence standard techniques (center manifold theorem) apply to reduce the bifurcation problem to one in a finite dimensional space (irrep of $G_\Gamma$).

For a given $\Gamma$ the area of a fundamental region is fixed (by Gauss-Bonnet formula) $\rightarrow$ no scale equivalence between lattices as in Euclidean plane. There are an infinite number of lattices in $D$.
Let $\Gamma \subset SU(1, 1)$ be a discrete subgroup which tiles $\mathbb{D}$ from a compact fundamental domain $F_\Gamma$ (polygon). Then $\Gamma$ is spanned by a finite number of hyperbolic boosts. $\Gamma$ is called a cocompact Fuchsian group (or a lattice group).
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Periodic patterns with a regular octagonal domain $F_T$

This is the simplest example of a lattice on $\mathbb{D}$.

- The regular octagonal lattice group $\Gamma$ is generated by four hyperbolic boosts.
- Vertex angles $\pi/8$, area $4\pi$.
- $\mathbb{D}/\Gamma \simeq$ double torus (genus 2).
- $G_\Gamma = G_0 \cup \kappa G_0$ where $\kappa : z \rightarrow \bar{z}$ and $G_0 \simeq GL(2, 3)$ ($|G_0| = 48$).
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- 13 irreducible representations of $G_\Gamma \rightarrow$ 13 different bifurcation problems: 4 with dim 1, 2 with dim 2, 4 with dim 3 and 3 with dim 4.
- All "generic" bifurcating patterns have been described in Faye & C. 2011.
An example with a 1-dim. representation of $G_\Gamma$

This is the axis of $\Gamma$-periodic states which are invariant under the 48-element subgroup of $G_\Gamma$ generated by $SL(2, 3) = \{ g \in GL(2, 3) \mid \det(g) = 1 \}$ and a reflection. Bifurcation is pitchfork.
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**Remark:** the numerical computation of $\Gamma$-periodic hyperbolic harmonics is tricky. There is no explicit formula (unlike in Euclidean case). Need to decompose $F_{\Gamma}$ in fundamental triangles tiling it by reflections, then apply finite elements numerical schemes.

Pascal Chossat*