Applications of Lie systems theory in classical and quantum physics

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Abstract

After a quick presentation of the theory of Lie systems from a geometric perspective, recent progresses on their applications when compatible geometric structures exist will be described, with an special emphasis in the particular case of admissible Kähler structures, and therefore with applications in Quantum Mechanics. More specifically it will be shown that they are useful in the study of the time evolution of a quantum system, as well as in particular cases of time-independent Schrödinger equation. Applications in control theory will also be exhibited

Outline

Introduction

Solution of systems of differential equations appearing in many physical problems is not an easy task. In geometric terms they are represented by vector fields, whose integral curves are the solutions of the system.

In order to find their solutions, i.e. the flow of the vector fields, as in a generic case there is not way of writing them in an explicit way, i.e. using fundamental functions, we are happy if, at least, we can express the solutions in terms of quadratures.

Characterisation of system admitting such type of solutions has received a lot of attention, and the answer is always based on the use of Lie algebras of vector fields containing the given one.

In general, in order to study such systems use is made of symmetry and reduction techniques. In such procedures the knowledge of some particular solutions of them or related systems may be useful.

For instance as far as Riccati equation is concerned one knows that

- ☐ If one particular solution is known, we can find the general solution by means of two quadratures.
- ☐ If two particular solutions are known, we can find the general solution by means of just one quadrature.
- ☐ If three particular solutions are known we can explicitly write the solution without any quadrature.

We are now interested in this particular kind of systems: the so called Lie systems.

They appear very often in many problems in science and engineering.

We will fix our attention to the particular case of quantum mechanics, where they are useful in:

- Studying the time evolution of a quantum system
- In particular cases of time-independent Schrödinger equation

Lie–Scheffers systems: a quick review

Lie–Scheffers systems $=$ Non-autonomous systems of n first-order differential equations admitting a ...

Superposition rule: a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}, \quad x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ being a generic set of particular solutions of the system and where $k_1, \ldots, k_n$ are real numbers.

They are a generalisation of linear superposition rules for homogeneous linear systems for which $m = n$ and $x = \Phi(x_{(1)}, \ldots, x_{(n)}; k_1, \ldots, k_n) = k_1\, x_{(1)} + \cdots + k_n\, x_{(n)}$ but

i) The number m may be different from the dimension n.

ii) The function Φ is nonlinear in this more general case.

They appear quite often in many different branches of science ranging from pure mathematics to classical and quantum physics, control theory, economy, etc. Forgotten for a long time they had a revival due to the work of Winternitz and coworkers.

One particular example is Riccati equation, of a fundamental importance in physics (for instance factorisation of second order differential operators, Darboux transformations and in general Supersymmetry in Quantum Mechanics) and in mathematics.

These systems are related with equations in Lie groups and in general connections in fibre bundles.

In the solution of such non-autonomous systems of first-order differential equations we can use techniques imported from group theory, for instance Wei–Norman method, and reduction techniques coming from the theory of connections.

Recent generalisations have also been shown to be useful for dealing with other systems of differential equations (e.g. Emden–Fowler equations, Abel equations).

The existence of additional compatible geometric structures, like symplectic or Poisson structures may be useful in the search for solutions.

Theorem: *Given a non-autonomous system of n first order differential equations*

$$\frac{dx^i}{dt} = X^i(x^1, \ldots, x^n, t), \quad i = 1 \ldots, n,$$

a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n, \quad x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, *such that the general solution is*

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n)\ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ *being a set of particular solutions of the system and where* $k_1, \ldots, k_n$, *are n arbitrary constants, is that the system can be written as*

$$\frac{dx^i}{dt} = b_1(t)\xi_1^i(x) + \cdots + b_r(t)\xi_r^i(x), \qquad i = 1 \ldots, n,$$

where $b_1, \ldots, b_r$, *are r functions depending only on t and* ξ_α^i, $\alpha = 1, \ldots, r$, *are functions of* $x = (x^1, \ldots, x^n)$, *such that the r vector fields in* $\mathbb{R}^n$ *given by*

$$X_\alpha \equiv \sum_{i=1}^n \xi_\alpha^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}\ , \qquad \alpha = 1, \ldots, r,$$

close on a real finite-dimensional Lie algebra, i.e. the X_α are l.i. and there are r^3 real numbers, $c_{\alpha\beta}{}^\gamma$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma \ .$$

The number r satisfies $r \leq mn$.

The geometric concept of superposition rule is the following:

A superposition rule for a t-dependent vector field X in a n-dimensional manifold M is a map $\Phi : M^m \times M \to M$ such that if $\{x_{(1)}(t), \ldots, x_{(m)}(t)\}$ is a generic set of integral curves of X, then $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t), k)$, with $k \in M$, is also integral curve of X, and each integral curve is obtained in this way.

The result of the Theorem in modern terms is that a t-dependent vector field X admits a superposition rule if there exist r fields $X_1, \ldots, X_r$ in M and functions $b_1(t), \ldots, b_r(t)$ such that $X(x,t)$ be a linear combination

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x).$$

The t-dependent vector field can be seen as a family of vector fields $\{X_t \mid t \in \mathbb{R}\}$.

Definition. *The minimal Lie algebra of a given a t-dependent vector field X on a manifold M is the smallest real Lie algebra, V^X, containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \mathrm{Lie}(\{X_t \mid t \in \mathbb{R}\})$.*

Definition. *The vector field associated to a non-autonomous system X allows us to define a generalised distribution $\mathcal{D}^X : x \in M \mapsto \mathcal{D}^X_x \subset TM$, where $\mathcal{D}_x = \{Y_x \mid Y \in V^X\} \subset T_xM$, and X also gives rise to a generalised co-distribution $\mathcal{V} : x \in M \mapsto \mathcal{V}_x \subset T^*M$, where $\mathcal{V}_x = \{\omega_x \mid \omega_x(Y_x) = 0, \forall Y_x \in \mathcal{D}^X_x\}$.*

Remark that the Lie–Scheffers theorem can be reformulated as follows:

Theorem: *A system X admits a superposition rule if and only if the minimal Lie algebra V^X is finite-dimensional.*

Definition. *A function $f : U \subset U^X \to \mathbb{R}$ is a local first integral (or t-independent constant of the motion) for a given t-dependent vector field X over $\mathbb{R}^n$ if $Xf = 0$*

Then f is a first integral if and only if $df \in \mathcal{V}^X|_U$.

One can easily prove that:

Property. *Given a t-dependent vector field X on a n-dimensional manifold M and a point $x \in U^X$ where the rank of $\mathcal{D}^X$ is equal to k, the associated co-distribution $\mathcal{V}^X$ admits, in a neighbourhood of x, a local basis of the form, $df_1, \ldots, df_{n-k}$, where, $f_1, \ldots, f_{n-k}$, is a family of fist integrals of X. Additionally, the space $\mathcal{I}^X|_U$ of first-integrals of the system X over an open U of M, can be put in the form*

$$\mathcal{I}^X|_U = \{g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{n-k} \to \mathbb{R},\ g = F(f_1, \ldots, f_{n-k})\}.$$

There exist different procedures to derive superposition rules for Lie systems. We can use a method based on the *diagonal prolongation* notion.

Definition. *Given a t-dependent vector field X over M, its diagonal prolongation to M^{m+1} is the t-dependent vector field $\widetilde{X}$ over M^{m+1} such that*

- $\widetilde{X}$ *projects onto X by the map* $\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in M^{m+1} \mapsto x_{(0)} \in M$, *that is,* $\mathrm{pr}_* \widetilde{X} = X$.
- $\widetilde{X}$ *is invariant under permutation* $x_{(i)} \leftrightarrow x_{(j)}$, *with* $i, j = 0, \ldots, m$.

The procedure to determine superposition rules described is:

i) Take a basis $X_1, \ldots, X_r$ of the Vessiot–Guldberg Lie algebra V associated with the Lie system.

ii) Choose the minimum integer m such that the diagonal prolongations to M^m of the elements of the previous basis are linearly independent at a generic point.

ii) Obtain n common first-integrals for the diagonal prolongations, $\widetilde{X}_1, \ldots, \widetilde{X}_r$, to M^{m+1} (for instance, by means of *the method of characteristics*).

iii) Obtain the expression of the variables of one of the spaces M only in terms of the other variables of M^{m+1} and the above mentioned n first-integrals.

The so obtained expressions give rise to a superposition rule in terms of any generic family of m particular solutions and n constants corresponding to the possible values of the derived first-integrals.

Some particular examples

A) Inhomogeneous linear systems:

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \ , \qquad i = 1, \dots, n.$$

The time-dependent vector field is

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \right) \frac{\partial}{\partial x^i} \ ,$$

which is a linear combination with t-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, Y_{ij} + \sum_{i=1}^{n} B^i(t)\, Y_i \ ,$$

of the $n^2 + n$ vector fields

$$Y_{ij} = x^j \frac{\partial}{\partial x^i}, \qquad Y_i = \frac{\partial}{\partial x^i} \ , \qquad i, j = 1, \dots, n \, .$$

These last vector fields have the following commutation relations:

$$[Y_i, Y_k] = 0 \;, \qquad [Y_{ij}, Y_k] = -\delta_{kj}\, Y_i \;, \qquad \forall\, i, j, k = 1, \dots, n\,.$$

• The set $\{Y_i \mid i = 1, \dots, n\}$ generates an Abelian ideal.

• The set $\{Y_{ij} \mid i, j = 1, \dots, n\}$ generates a Lie subalgebra.

• The Vessiot Lie algebra is isomorphic to the $(n^2 + n)$-dimensional Lie algebra of the affine group.

In this case $r = n^2 + n$ and using the general theory one can see that $m = n + 1$ and the equality $r = m\, n$ also follows.

The superposition function $\Phi : \mathbb{R}^{n(n+1)} \to \mathbb{R}^n$ is:

$$x = \Phi(u_1, \dots, u_{n+1}; k_1, \dots, k_n) = u_1 + k_1(u_2 - u_1) + \cdots + k_n(u_{n+1} - u_1),$$

i.e. the general solution can be written in terms of $n + 1$ generic solutions as:

$$\Phi(x_{(1)}, \dots, x_{(n+1)}; k_1, \dots, k_n) = x_{(1)} + k_1(x_{(2)} - x_{(1)}) + \cdots + k_n(x_{(n+1)} - x_{(1)}).$$

B) The Riccati equation ($n = 1$)

$$\frac{dx(t)}{dt} = a_2(t)\, x^2(t) + a_1(t)\, x(t) + a_0(t)\ .$$

Now $m = r = 3$ and the superposition principle comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k\ ,$$

or in other words,

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) + k\, x_2(t)(x_1(t) - x_3(t))}{(x_3(t) - x_2(t)) + k\,(x_1(t) - x_3(t))}\ ,$$

i.e. the superposition rule involves three different solutions, $m = 3$. The value $k = \infty$ must be accepted, otherwise we do not obtain the solution x_2.

The vector fields $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ are given by

$$Y^{(1)} = \frac{\partial}{\partial x}\,, \quad Y^{(2)} = x\,\frac{\partial}{\partial x}\,, \quad Y^{(3)} = x^2\,\frac{\partial}{\partial x}\,,$$

that close on a three-dimensional real Lie algebra, i.e. $r = 3$, with defining relations

$$[Y^{(1)}, Y^{(2)}] = Y^{(1)}\,, \quad [Y^{(1)}, Y^{(3)}] = 2Y^{(2)}\,, \quad [Y^{(2)}, Y^{(3)}] = Y^{(3)}\,.$$

Then, the associated Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$.

C) Lie–Scheffers systems on Lie groups

Consider the set of right–invariant vector fields in G spanning the opposite of the Lie algebra $\mathfrak{g}$. Similarly can be done with the set left–invariant vector fields in G, i.e. the Lie algebra $\mathfrak{g}$.

If $\{a_1,\ldots,a_r\}$ is a basis for the tangent space T_eG and X^R_α denotes the right-invariant vector field in G such that $X^R_\alpha(e)=a_\alpha$, a Lie–Scheffers system is

$$\dot g(t)=-\sum_{\alpha=1}^r b_\alpha(t)X^R_\alpha(g(t))\ .$$

When applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides we obtain the equation on T_eG

$$(R_{g(t)^{-1}})_{*g(t)}(\dot g(t))=-\sum_{\alpha=1}^r b_\alpha(t)a_\alpha\ , \qquad (**)$$

This is usually written with a slight abuse of notation:

$$(\dot g\, g^{-1})(t)=-\sum_{\alpha=1}^r b_\alpha(t)a_\alpha\ .$$

Such equation is right-invariant. Then,

If $\bar{g}(t)$ is a solution of (**) with initial condition $\bar{g}(0) = e$, the solution $g(t)$ with initial conditions $g(0) = g_0$ is given by $\bar{g}(t)g_0$.

Moreover, there is a superposition rule $\Phi : G \times G \to G$ involving one solution. If g_p is a particular solution, then its general solution is $g(t) = g_p(t)\, g_0$, i.e., there is a superposition rule

$$\Phi(g, g_0) = g\, g_0.$$

This example is very useful because there are many other examples related with them as explained next.

D) Lie-Scheffers systems on homogeneous spaces for Lie groups

Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. If $\tau : G \to G/H$ is the natural projection, the right–invariant vector fields X_α^R are τ-projectable and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M,

$$\tau_{*g} X_\alpha^R(g) = -X_\alpha(gH) \ ,$$

and we will have an associated Lie–Scheffers system on M with Vessiot algebra $\mathfrak{g}$:

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x) \ .$$

Therefore, a solution of this last system starting from x_0 will be:

$$x(t) = \Phi(g(t), x_0) \ ,$$

with $g(t)$ being a solution of (**) starting from $e \in G$, $g(0) = e$.

The converse property is true: Given a Lie Scheffers system defined by complete vector fields with associated Lie algebra $\mathfrak{g}$, we can see these as fundamental vector fields relative to an action which can be found by integrating the vector fields.

Wei-Norman method

Let G be a r–dimensional Lie group and $\{a_1, \ldots, a_r\}$ a basis of T_eG, consider the equation determining the curves $g(t) \in G$ such that

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \in T_eG \ ,$$

with $g(0) = e \in G$.

In order to solve directly such equation we can use a method which is a generalisation of the one proposed by Wei and Norman for finding the time evolution operator for a linear systems of type

$$\mathrm{i}\, \frac{dU(t)}{dt} = H(t)U(t) \ ,$$

with $U(0) = I$.

Remark that there exist alternative methods for solving the equation by reducing the problem to a simpler one.

Property: If $g(t)$, $g_1(t)$ and $g_2(t)$ are differentiable curves in G such that $g(t) = g_1(t)g_2(t)$, $\forall t \in \mathbb{R}$, then,

$$
\begin{aligned}
R_{g(t)^{-1}\,*g(t)}(\dot g(t)) &= R_{g_1(t)^{-1}\,*g_1(t)}(\dot g_1(t)) \\
&+ \mathrm{Ad}\,(g_1(t))\left\{R_{g_2(t)^{-1}\,*g_2(t)}(\dot g_2(t))\right\}\,.
\end{aligned}
$$

The generalisation to several factors is as follows:

If $g(t) = g_1(t)g_2(t)\cdots g_l(t) = \prod_{i=1}^{l} g_i(t)$, then we have:

$$
R_{g(t)^{-1}\,*g(t)}(\dot g(t)) = \sum_{i=1}^{l}\left(\prod_{j<i}\mathrm{Ad}\,(g_j(t))\right)\left\{R_{g_i(t)^{-1}\,*g_i(t)}(\dot g_i(t))\right\}\,,
$$

where it has been taken $g_0(t) = e$ for all t.

The generalized Wei–Norman method consists on writing $g(t)$ in terms of its second kind canonical coordinates,

$$
g(t) = \prod_{\alpha=1}^{r}\exp(-v_\alpha(t)a_\alpha) = \exp(-v_1(t)a_1)\cdots\exp(-v_r(t)a_r)\,,
$$

and transforming the equation into a differential equation system for the $v_\alpha(t)$, with initial conditions $v_\alpha(0) = 0$ for all $\alpha = 1, \ldots, r$.

Then, using the expression of the above property, with $l = r = \dim G$ and $g_\alpha(t) = \exp(-v_\alpha(t) a_\alpha)$ for all α, we see that

$$\begin{aligned} R_{g(t)^{-1}\,*g(t)}(\dot g(t)) &= -\sum_{\alpha=1}^{r} \dot v_\alpha \left(\prod\nolimits_{\beta<\alpha} \mathrm{Ad}\,(\exp(-v_\beta(t) a_\beta))\right) a_\alpha \\ &= -\sum_{\alpha=1}^{r} \dot v_\alpha \left(\prod\nolimits_{\beta<\alpha} \exp(-v_\beta(t) \mathrm{Ad}\,(a_\beta))\right) a_\alpha \,. \end{aligned}$$

Then, the fundamental expression of the Wei–Norman method is:

$$\sum_{\alpha=1}^{r} \dot v_\alpha \left(\prod_{\beta<\alpha} \exp(-v_\beta(t) \mathrm{Ad}\,(a_\beta))\right) a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \,,$$

with $v_\alpha(0) = 0$, $\alpha = 1, \ldots, r$.

The resulting system of differential equations for the functions $v_\alpha(t)$ is integrable by quadratures if the Lie algebra is solvable, and in particular, for nilpotent Lie algebras.

The reduction method

The important ingredient is an equation on a Lie group

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\, a_\alpha \in T_eG\ , \qquad (\bullet)$$

with $g(0) = e \in G$, which may be solved by a method extending the so called Wei–Norman method.

It may happen that the only different from zero coefficients b_α are those corresponding to those a_α of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the equation reduces to a simpler equation on a Lie subgroup, involving less coordinates.

The fundamental result is that if we know a particular solution of the problem associated in a homogeneous space, the original solution reduces to one on the subgroup.

Let us choose a curve $g'(t)$ in the group G, and define the curve $\overline{g}(t)$ by $\overline{g}(t) =$

$g'(t)g(t)$. Then if $g(t)$ is a solution of $(\bullet)$, the new curve in G, $\overline{g}(t)$, is solution of a new Lie system.

Indeed,

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}\,(g'(t))a_\alpha \;,$$

which leads to an equation similar to the original one but with a different right hand side.

In this way we can define an action of the group of curves in the Lie group G on the set of Lie systems on the group. This can be used to reduce a given Lie system to a simpler one.

The aim is to choose the curve $g'(t)$ in such a way that the new equation be simpler. For instance, we can choose a subgroup H and look for a choice of $g'(t)$ such that the right hand side lies in T_eH, and hence $\overline{g}(t) \in H$ for all t.

If $\Psi : G \times M \to M$ is a transitive action of G on a homogeneous space M, which can be identified with the set G/H of left-cosets, by choosing a fixed point x_0, then the integral curves starting from the point x_0 associated to both Lie systems are related by

$$\overline{x}(t) = \Psi(\overline{g}(t), x_0) = \Psi(g'(t)g(t), x_0) = \Psi(g'(t), x(t)) \ .$$

Therefore, this gives an action of the group of curves in G on the set of associated Lie systems in homogeneous spaces.

More explicitly, if we consider ta curve $g'(t)$ in the group, the Lie system transforms into a new one

$$\dot{\bar{x}} = \sum_{\alpha=1}^{r} \bar{b}_\alpha(t) X_\alpha(\bar{x}) \ ,$$

in which

$$\bar{b} = \mathrm{Ad}\,(g'(t))b(t) + \dot{g}'\, g'^{-1} \ .$$

The important result is that the knowledge of a particular solution of the associated Lie system in G/H allows us to reduce the problem to one in the subgroup H.

Theorem: *Each solution of (•) on the group G can be written in the form $g(t) = g_1(t)\, h(t)$, where $g_1(t)$ is a curve on G projecting onto a solution $\tilde{g}_1(t)$ for* *the left action λ of G on the homogeneous space G/H and $h(t)$ is a solution of an equation but for the subgroup H, explicitly given by*

$$(\dot{h}\,h^{-1})(t) = -\mathrm{Ad}\,(g_1^{-1}(t))\left(\sum_{\alpha=1}^{r} b_\alpha(t)a_\alpha + (\dot{g}_1\,g_1^{-1})(t)\right) \in T_eH \ .$$

Lie sytems in control theory

Control systems are described by systems of differential equations

$$\frac{dx^i}{dt} = X^i(x^j, u^\alpha), \quad i, j = 1, \ldots, n, \quad \alpha = 1, \ldots, m,$$

with u^α, for $\alpha = 1, \ldots, r$, being control functions of t, which are to be determined in such a way that the trajectory passes through one or two specific points in the configuration space, or maybe gives some cost functional a stationary value.

From a geometric point of view, the framework of a control system is a bundle B (usually a trivial vector bundle) on the state space manifold M, with projection $\pi_B : B \to M$. The control dynamical system corresponds then to the integral curves of a vector field along the projection π_B, which in local coordinates, (x^i, u_α), reads

$$\dot{x}^i = X^i(x, u)\,, \quad x \in M\,, \quad u \in B_x = \pi_B^{-1}(x)\ .$$

A special class is made of control systems which are affine in the controls:

$$\dot{x}^i = X_0^i(x) + \sum_{\alpha=1}^{r} u_\alpha\, X_\alpha^i(x)\ , \qquad i = 1, \ldots, n = \dim M\ .$$

In principle, the control functions u_α are supposed to depend on time (in the control theory terminology the system is then operating in *open loop*).

The vector field along π_B is $X = X_0 + \sum_{\alpha=1}^r u_\alpha\, X_\alpha^i$ where $X_0 = \sum_{i=1}^n X_0^i\, \partial/\partial x^i \in \mathfrak{X}(M)$ and $X_\alpha = \sum_{i=1}^n X_\alpha^i\, \partial/\partial x^i \in \mathfrak{X}(M)$.

If X is drift-less, i.e., $X_0 = 0$, then it is a Lie system when the set of vector fields $\{X_\alpha^i \partial/\partial x^i\}$ closes on a finite-dimensional real Lie algebra.

It may also happen that starting with a control system that is of a similar form, but where the vector fields $\{X_\alpha^i \partial/\partial x^i\}$ do not close on a Lie algebra, an appropriate *feedback transformation* $u_\alpha(x,t) = \sum_{\beta=1}^r f_{\alpha\beta}(x) v_\beta(t)$ could lead to a new system written as

$$\dot{x}^i = \sum_{\alpha=1}^r v_\alpha(t)\, Y_\alpha^i(x)\ , \qquad i = 1, \ldots, n = \dim M\ ,$$

where $Y_\alpha(x) = \sum_{i=1}^r f_{\beta\alpha}(x) X_\beta(x)$ do close a finite-dimensional Lie algebra, i.e. giving rise to a Lie system.

A control system is said to be controllable if for any given initial point p there exists an integral curve of the corresponding vector field along $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ such that $(\pi \circ \gamma)(0) = p$ and a value t_1 of the parameter of the curve γ such that $(\pi \circ \gamma)(t_1) = q$.

Lie systems arise as control systems for which

$$X(t, x) = u^1(t) X_1(x) + \cdots + u^r(t) X_r(x)$$

with the vector fields X_α closing on a finite-dimensional real Lie algebra. They are usually called drift-free systems, linear in the control functions $u^\alpha(t)$.

We know that the cases in which the X_α are either vector fields in Lie groups G, or vector fields in a homogeneous space for G can be dealt with according to the theory explained before: reducing the problem to solve an equation on the group G.

$$\dot{g}(t) = \sum_{\alpha=1}^{r} u_\alpha(t) X_\alpha(g(t)) \; .$$

When the vector fields X_α in the Lie group G are right- or left-invariant the control system is said to be right- or left-invariant.

Controllability of control systems on Lie groups has been analyzed by Brockett and Jurdjevic and Sussmann. It can be determined by studying algebraic properties of the corresponding Lie algebra $\mathfrak{g}$.

Theorem: *A drift-free right invariant system on a connected Lie group G is controllable if and only if the Lie algebra generated by $\{X_1, \ldots, X_r\}$ is $\mathfrak{g}$.*

Proof: If $\mathfrak{h}$ is the Lie algebra generated by $\{X_1, \ldots, X_r\}$, then the Lie algebra of the Lie system is not $\mathfrak{g}$ but the subalgebra $\mathfrak{h}$. The orbit of the neutral element $e \in G$ is the subgroup H of G with Lie subalgebra $\mathfrak{h}$. It is then clear that if $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{g}$, the system is not controllable, while it is so when $\mathfrak{h} = \mathfrak{g}$.

Example:

A simplified model of maneuvering an automobile or, essentially the same control system, the problem of path planning for a robot unicycle.

The configuration space is $\mathbb{R}^2 \times S^1$, with coordinates (x_1, x_2, x_3). The control system can be written as

$$\dot{x}_1 = b_2(t) \sin x_3 \,, \quad \dot{x}_2 = b_2(t) \cos x_3 \,, \quad \dot{x}_3 = b_1(t) \,,$$

giving the integral curves of $b_1(t)\, X_1(x) + b_2(t)\, X_2(x)$, where

$$X_1 = \frac{\partial}{\partial x_3} \,, \qquad X_2 = \sin x_3 \frac{\partial}{\partial x_1} + \cos x_3 \frac{\partial}{\partial x_2} \,.$$

The Lie bracket of both vector fields,

$$X_3 = [X_1,\, X_2] = \cos x_3 \frac{\partial}{\partial x_1} - \sin x_3 \frac{\partial}{\partial x_2}$$

is linearly independent from X_1, X_2. They satisfy

$$[X_1,\, X_2] = X_3 \,, \quad [X_2,\, X_3] = 0 \,, \quad [X_1,\, X_3] = -X_2 \,,$$

therefore closing on a Lie algebra isomorphic to $\mathfrak{se}(2)$.

The SODE Lie systems

A system of second order differential equations

$$\ddot{x}^i = f^i(t, x, \dot{x}), \qquad i = 1, \ldots, n,$$

can be studied through the corresponding system of first order differential equations

$$\begin{cases} \dfrac{dx^i}{dt} & = & v^i \\ \dfrac{dv^i}{dt} & = & f^i(t, x, v) \end{cases}$$

with associated t-dependent vector field

$$X = v^i \frac{\partial}{\partial x^i} + f^i(t, x, v) \frac{\partial}{\partial v^i}$$

We call SODE Lie systems those for which X is a Lie system, i.e. it can be written as a linear combination with t-dependent coefficients of vector fields closing a finite-dimensional real Lie algebra.

Examples

A) The 1-dim harmonic oscillator with time-dependent frequency

The equation of motion is

$$\ddot{x} = -\omega^2(t)x,$$

with associated system

$$\left\{ \begin{array}{rcl} \dot{x} & = & v \\ \dot{v} & = & -\omega^2(t)x \end{array} \right.$$

and vector field

$$X = v\frac{\partial}{\partial x} - \omega^2(t)x\,\frac{\partial}{\partial v}\ ,$$

which is a linear combination $X = X_1 - \omega^2(t)X_2$ with

$$X_1 = v\frac{\partial}{\partial x}\,, \qquad X_2 = x\,\frac{\partial}{\partial v},$$

such that if

$$X_3 = \frac{1}{2}\left(v\frac{\partial}{\partial v} - x\frac{\partial}{\partial x}\right)\ ,$$

then

$$[X_1, X_2] = 2\,X_3\,, \quad [X_1, X_3] = -X_1\,, \quad [X_2, X_3] = X_2\,,$$

a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. This system has no first integrals.

B) **The 2-dim isotropic harmonic oscillator with time-dependent frequency**

The equation of motion is

$$\left\{\begin{array}{rcl} \ddot{x}_1 &=& -\omega^2(t)x_1 \\ \ddot{x}_2 &=& -\omega^2(t)x_2 \end{array}\right.$$

with associated system

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& v_1 \\ \dot{v}_1 &=& -\omega^2(t)x_1 \\ \dot{x}_2 &=& v_2 \\ \dot{v}_2 &=& -\omega^2(t)x_2 \end{array}\right.$$

and the t-dependent vector field

$$X = v_1\frac{\partial}{\partial x_1} - \omega^2(t)x_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial x_2} - \omega^2(t)x_2\frac{\partial}{\partial v_2} \;,$$

is a linear combination $X = X_1 - \omega^2(t)X_2$ with

$$X_1 = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}\,, \qquad X_2 = x_1\frac{\partial}{\partial v_1} + x_2\frac{\partial}{\partial v_2}\,,$$

such that if

$$X_3 = \frac{1}{2}\left(v_1\frac{\partial}{\partial v_1} - x_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial v_2} - x_2\frac{\partial}{\partial x_2}\right)\,,$$

then

$$[X_1, X_2] = 2\,X_3\,, \quad [X_1, X_3] = -X_1\,, \quad [X_2, X_3] = X_2\,,$$

once again a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The system admits an invariant because, if F is given by $F(x_1, x_2, v_1, v_2)$, then $X_1 F = 0$ shows that there exists a function $\bar{F}(\xi, v_1, v_2)$ with $\xi = x_1 v_2 - x_2 v_1$, such that $F(x_1, x_2, v_1, v_2) = \bar{F}(\xi, v_1, v_2)$ while the second condition

$$x_1\,\frac{\partial \bar{F}}{\partial v_1} + x_2\,\frac{\partial \bar{F}}{\partial v_2} = 0$$

i.e. we obtain the first integral

$$F = x_1 v_2 - x_2 v_1,$$

which can be seen as a partial superposition rule.

With three copies of the same harmonic oscillator, the vector fields X_1 and X_2 are

$$X_1 = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v \frac{\partial}{\partial x}\,, \qquad X_2 = x_1 \frac{\partial}{\partial v_1} + x_2 \frac{\partial}{\partial v_2} + x \frac{\partial}{\partial v}\,,$$

which determine the first integrals F as solutions of $X_1 F = X_2 F = 0$. The condition $X_1 F = 0$ says that there exists a function $\bar{F} : \mathbb{R}^5 \to \mathbb{R}^2$ such that $F(x_1, x_2, x, v_1, v_2, v) = \bar{F}(\xi_1, \xi_2, v_1, v_2, v)$ with $\psi_1(x_1, x_2, x, v_1, v_2, v) = x v_1 - x_1 v$ and $\psi_2(x_1, x_2, x, v_1, v_2, v) = x v_2 - x_2 v$, and the condition $X_2 F = 0$ transforms into

$$x_1 \frac{\partial \bar{F}}{\partial v_1} + x_2 \frac{\partial \bar{F}}{\partial v_2} + x \frac{\partial \bar{F}}{\partial v} = 0,$$

i.e. ξ_1 and ξ_2 are first integrals. They produce a superposition rule, because

$$\left\{ \begin{array}{ccc} x v_2 - x_2 v & = & k_1 \\ x_1 v - v_1 x & = & k_2 \end{array} \right. \Longleftrightarrow \begin{pmatrix} v_2 & -x_2 \\ -v_1 & x_1 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

from where we obtain the expected superposition rule:

$$x = C_1\, x_1 + C_2\, x_2\,, \qquad v = C_1\, v_1 + C_2\, v_2\,, \qquad C_i = \frac{k_i}{x_1 v_2 - x_2 v_1}.$$

C) **Pinney equation**:

The Pinney equation is the following second order non-linear differential equation:

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3}\,,$$

where k is a constant. The corresponding system of first order differential eqns is

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -\omega^2(t)x + \dfrac{k}{x^3} \end{cases}$$

and the associated t-dependent vector field

$$X = v\frac{\partial}{\partial x} + \left(-\omega^2(t)x + \frac{k}{x^3}\right)\frac{\partial}{\partial v}\,.$$

This is a Lie system because it can be written as

$$X = L_2 - \omega^2(t)L_1\,,$$

where:

$$L_1 := x\frac{\partial}{\partial v}, \quad L_2 = \frac{k}{x^3}\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}.$$

The vector fields L_1 and L_2 span a three-dimensional real Lie algebra $\mathfrak{g}$ with nonzero defining relations:

$$[L_1, L_2] = 2L_3, \quad [L_3, L_2] = -L_2, \quad [L_3, L_1] = L_1$$

where

$$L_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right),$$

which is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The fact that they have the same associated Lie algebra means that they can be solved simultaneously in the group $SL(2, \mathbb{R})$ by the equation

$$\dot{g}\, g^{-1} = \omega^2(t)\, a_1 - a_2$$

Note that this isotonic oscillator shares with the harmonic one the property of having a period independent of the energy, i.e. they are isochronous, and in the quantum case they have a equispaced spectrum.

D) **Ermakov system**

Consider the system

$$\begin{cases} \dot{x} &= v_x \\ \dot{v}_x &= -\omega^2(t)x \\ \dot{y} &= v_y \\ \dot{v}_y &= -\omega^2(t)y + \dfrac{1}{y^3} \end{cases}$$

with associated vector field

$$X = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} - \omega^2(t)x\frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3}\right)\frac{\partial}{\partial v_y}\,,$$

which is a linear combination with time-dependent coefficients, $X = -\omega^2(t)X_1 + X_2$, of the vector fields

$$X_1 = x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y}\,, \qquad X_2 = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + \frac{1}{y^3}\frac{\partial}{\partial v_y}\,.$$

This system is made up by two Lie systems closing on a $\mathfrak{sl}(2,\mathbb{R})$ algebra, and the full system is also a Lie systems because the vector fields X_1 and X_2 span a three-dimensional real Lie algebra $\mathfrak{g}$ with nonzero defining relations:

$$[X_1, X_2] = 2X_3, \quad [X_3, X_2] = -X_2, \quad [X_3, X_1] = X_1$$

where

$$X_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right) \, ,$$

The second subsystem of first order differential equations is usually called Pinney equation. The generators of the Lie system with algebra $\mathfrak{sl}(2,\mathbb{R})$ span a distribution of dimension two and there is no first integral of the motion for such subsystem.

By adding the other $\mathfrak{sl}(2,\mathbb{R})$ linear Lie system, the h.o. with time dependent angular frequency, as the distribution in the four-dimensional space is of rank three there is an integral of motion, which can be obtained from $X_1F = X_2F = 0$. But X_1F means that $F(x,y,v_x,v_y) = \bar{F}(x,y,\xi)$ with $\xi = xv_y - yv_x$, and then $X_2F = 0$ is written

$$v_x\frac{\partial \bar{F}}{\partial x} + v_x\frac{\partial \bar{F}}{\partial x} + \frac{x}{y^3}\frac{\partial \bar{F}}{\partial \xi}$$

and the associated characteristics system we obtain

$$\frac{x\,dy - y\,dx}{\xi} = \frac{y^3\,d\xi}{x} \Longrightarrow \frac{d(x/y)}{\xi} + \frac{y\,d\xi}{x} = 0$$

from where the well-known Ermakov invariant is found:

$$\psi(x,y,v_x,v_y) = \left(\frac{x}{y}\right)^2 + \xi^2 = \left(\frac{x}{y}\right)^2 + (xv_y - yv_x)^2.$$

E) Generalized Ermakov system

It is the system given by:

$$\begin{cases} \ddot{x} &= \dfrac{1}{x^3} f(y/x) - \omega^2(t) x \\ \ddot{y} &= \dfrac{1}{y^3} g(y/x) - \omega^2(t) y \end{cases}$$

In the particular case $f(u) = 0$ and $g(u) = 1$ reduces to the Ermakov system.

This system can be written as a first order one by doubling the number of degrees of freedom by introducing the new variables v_x and v_y:

$$\begin{cases} \dot{x} &= v_x \\ \dot{v}_x &= -\omega^2(t) x + \dfrac{1}{x^3} f(y/x) \\ \dot{y} &= v_y \\ \dot{v}_y &= -\omega^2(t) y + \dfrac{1}{y^3} g(y/x) \end{cases}$$

which determines the integral curves of the vector field

$$X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial v_y} + \left(-\omega^2(t) x + \frac{1}{x^3} f(y/x) \right) \frac{\partial}{\partial v_x} + \left(-\omega^2(t) y + \frac{1}{y^3} g(y/x) \right) \frac{\partial}{\partial v_y} \, .$$

Such vector field can be written as a linear combination

$$X = N_2 - \omega^2(t)\, N_1$$

where N_1 and N_2 are the vector fields

$$N_1 = x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y}, \quad N_2 = v_x\frac{\partial}{\partial x} + \frac{1}{x^3}f(y/x)\frac{\partial}{\partial v_x} + v_y\frac{\partial}{\partial y} + \frac{1}{y^3}g(y/x)\frac{\partial}{\partial v_y},$$

Note that these vector fields generate a three-dimensional real Lie algebra with a third generator

$$N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y}\right).$$

In fact, as

$$[N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_3, N_2] = -N_2$$

they generate a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Therefore the system is a Lie system.

There exists a first integral for the motion, $F:\mathbb{R}^4\to\mathbb{R}$, for any $\omega^2(t)$, because this Lie system has an associated integrable distribution of rank three and the manifold is four-dimensional.

This first integral F satifies $N_iF=0$ for $i=1,\ldots,3$, but as $[N_1,N_2]=2N_3$ it is enough to impose $N_1F=N_2F=0$. Then, if $N_1F=0$,

$$x\,\frac{\partial F}{\partial v_x}+y\,\frac{\partial F}{\partial v_y}=0\,,$$

and according to the method of characteristics we obtain:

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dv_x}{x}=\frac{dv_y}{y}$$

and therefore there exists a function $\bar F:\mathbb{R}^3\to\mathbb{R}$ such that $F(x,y,v_x,v_y)=\bar F(x,y,\xi=xv_y-yv_x)$. The condition $N_2F=0$ reads now

$$v_x\frac{\partial\bar F}{\partial x}+v_y\frac{\partial\bar F}{\partial y}+\left(-\frac{y}{x^3}f(y/x)+\frac{x}{y^3}g(y/x)\right)\frac{\partial\bar F}{\partial\xi}\,.$$

We can therefore consider the associated system the characteristics are given by:

$$\frac{dx}{v_x}=\frac{dy}{v_y}=\frac{d\xi}{-\frac{y}{x^3}f(y/x)+\frac{x}{y^3}g(y/x)}$$

But using that

$$\frac{-y\,dx + x\,dy}{\xi} = \frac{dx}{v_x} = \frac{dy}{v_y}$$

we arrive to

$$\frac{-y\,dx + x\,dy}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f(\frac{y}{x}) + \frac{x}{y^3} g(\frac{y}{x})}$$

i.e.

$$-\frac{y^2 d\left(\frac{x}{y}\right)}{\xi} = \frac{d\xi}{-\frac{y}{x^3} f(\frac{y}{x}) + \frac{x}{y^3} g(\frac{y}{x})}$$

and integrating we obtain the first integral

$$\frac{1}{2}\xi^2 + \int^u \left[\frac{1}{\zeta^3}\, f\left(\frac{1}{\zeta}\right) + \zeta\, g\left(\frac{1}{\zeta}\right)\right] d\zeta\,.$$

This first integral allows us to determine a solution of one subsystem in terms of a solution of the other equation.

F) The Pinney equation revisited

Consider the system of first order differential equations:

$$\begin{cases} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \\ \dot{v}_x &= -\omega^2(t)x \\ \dot{v}_y &= -\omega^2(t)y + \dfrac{k}{y^3} \\ \dot{v}_z &= -\omega^2(t)z \end{cases}$$

that corresponds to the vector field

$$X = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + \frac{k}{y^3}\frac{\partial}{\partial v_y} - \omega^2(t)\left(x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial v_z}\right),$$

which can be expressed as $X = N_2 - \omega^2(t)N_1$ where the vector fields N_1 and N_2 are:

$$N_1 = y\frac{\partial}{\partial v_y} + x\frac{\partial}{\partial v_x} + z\frac{\partial}{\partial v_z}, \quad N_2 = v_y\frac{\partial}{\partial y} + \frac{1}{y^3}\frac{\partial}{\partial v_y} + v_x\frac{\partial}{\partial x} + v_z\frac{\partial}{\partial z}.$$

These vector fields generate a three-dimensional real Lie algebra with the vector field

N_3 given by

$$N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial z} - v_z\frac{\partial}{\partial v_z}\right).$$

In fact, as

$$[N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_3, N_2] = -N_2,$$

they generate a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The system is a Lie system.

The distribution generated by these fundamental vector fields has rank three. Thus, as the manifold of the Lie system is of dimension six we obtain three time-independent integrals of motion.

- The Ermakov invariant I_1 of the subsystem involving variables x and y.
- The Ermakov invariant I_2 of the subsystem involving variables y and z.
- The Wronskian W of the subsystem involving variables x and z.

They define a foliation with three-dimensional leaves.

We can use this foliation for obtaining a superposition rule in terms of these three first integrals.

The Ermakov invariants read as:

$$\begin{aligned} I_1 &= \tfrac{1}{2}\left((yv_x - xv_y)^2 + k\left(\tfrac{x}{y}\right)^2\right) \\ I_2 &= \tfrac{1}{2}\left((yv_z - zv_y)^2 + k\left(\tfrac{z}{y}\right)^2\right) \end{aligned}$$

and W is:

$$W = x_1 v_{v_z} - z v_x .$$

In terms of these three integrals we can obtain an explicit expression of y in terms of x, z and the integrals I_1, I_2, W:

$$y = \frac{2}{W}\left(I_2 x^2 + I_1 z^2 \pm \sqrt{4I_1 I_2 - kW^2}\, xz\right)^{1/2} .$$

This can be interpreted as saying that there is a superposition rule allowing us to express the general solution of the Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with time-dependent frequency.

Structure preserving Lie systems

There are particularly interesting cases in which the manifold M is endowed with additional structures. For instance, let (M, Ω) be a symplectic manifold and the vector fields arising in the expression of the t-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra.

These vector fields correspond to a symplectic action of the Lie group G on (M, Ω).

The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = -dh_\alpha$, do not close on the same Lie algebra under Poisson bracket, but we can only say that

$$d\left(\{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]}\right) = 0 \ ,$$

and then they span a Lie algebra extension of the original one.

The important fact is that we can define a t-dependent Hamiltonian

$$h_t = \sum_\alpha b_\alpha(t)\, h_\alpha,$$

with the functions h_α closing a Lie algebra, in such a wat hat $i(X_t)\Omega = -dh_t$.

As an example we can consider the differential equation of an n-dimensional Winternitz–Smorodinsky oscillator of the form

$$\left\{\begin{array}{rcl} \dot{x}_i &=& p_i, \\ \dot{p}_i &=& -\omega^2(t)x_i + \dfrac{k}{x_i^3}, \end{array}\right. \qquad i = 1, \ldots, n.$$

which describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}^n$

$$X_t = \sum_{i=1}^n \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i + \frac{k}{x_i^3}\right)\frac{\partial}{\partial p_i}\right],$$

which can be written as $X_t = X_2 + \omega^2(t)X_1$ with X_1, X_2 and $X_3 = -[X_1, X_2]$ being given by

$$X_1 = -\sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, \quad X_2 = \sum_{i=1}^n \left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3}\frac{\partial}{\partial p_i}\right), \quad X_3 = \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i}\right).$$

Note that X_t is a Lie system, because X_1, X_2 and X_3 close on a $\mathfrak{sl}(2, \mathbb{R})$ algebra:

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = -X_2.$$

Moreover, the preceding vector fields are Hamiltonian vector fields with respect to the usual symplectic form $\omega_0 = \sum_{i=1}^{n} dx^i \wedge dp_i$ with Hamiltonian functions

$$h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad h_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \frac{k}{x_i^2}\right), \qquad h_3 = \sum_{i=1}^{n} x_i p_i,$$

which obey that

$$\{h_1, h_2\} = h_3, \qquad \{h_1, h_3\} = -h_1, \qquad \{h_2, h_3\} = h_2.$$

Consequently, every curve h_t taking values in the Lie algebra $(W, \{\cdot,\cdot\})$ spanned by h_1, h_2 and h_3 gives rise to a Lie system which is Hamiltonian in $\mathrm{T}^*\mathbb{R}^n$ with respect to the symplectic structure ω_0 in such a way that

$$X_t = X_2 + \omega^2(t) X_1 = \widehat{\omega}_0^{-1}(dh_2 + \omega^2(t) dh_1),$$

i.e. the Hamiltonian is $h_t = h_2 + \omega^2(t) h_1$.

We can go a step further and consider Lie systems in (may be degenerate) Poisson manifolds, or even more generally in Dirac manifolds.

Geometric approach to Quantum Mechanics

The Schrödinger picture of Quantum mechanics admits a geometric interpretation similar to that of classical mechanics.

A separable complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ can be considered as a real linear space, to be then denoted $\mathcal{H}_{\mathbb{R}}$. The norm in $\mathcal{H}$ defines a norm in $\mathcal{H}_{\mathbb{R}}$, where $\|\psi\|_{\mathbb{R}} = \|\psi\|_{\mathbb{C}}$.

The linear real space $\mathcal{H}_{\mathbb{R}}$ is endowed with a natural symplectic structure as follows:

$$\omega(\psi_1, \psi_2) = 2\,\mathrm{Imag}\,\langle\psi_1, \psi_2\rangle.$$

The Hilbert $\mathcal{H}_{\mathbb{R}}$ can be considered as a real manifold modelled by a Banach space admitting a global chart.

The tangent space $T_\phi\mathcal{H}_{\mathbb{R}}$ at any point $\phi \in \mathcal{H}_{\mathbb{R}}$ can be identified with $\mathcal{H}_{\mathbb{R}}$ itself: the isomorphism associates $\psi \in \mathcal{H}_{\mathbb{R}}$ with the vector $\dot\psi \in T_\phi\mathcal{H}_{\mathbb{R}}$ given by:

$$\dot\psi f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0}, \qquad \forall f \in C^\infty(\mathcal{H}_{\mathbb{R}}).$$

The real manifold can be endowed with a symplectic 2-form ω:

$$\omega_\phi(\dot{\psi}, \dot{\psi}') = 2\,\mathrm{Imag}\,\langle \psi, \psi' \rangle \ .$$

One can see that the constant symplectic structure ω in $\mathcal{H}_\mathbb{R}$, considered as a Banach manifold, is exact, i.e., there exists a 1-form $\theta \in \bigwedge^1(\mathcal{H}_\mathbb{R})$ such that $\omega = -d\theta$. Such a 1-form $\theta \in \bigwedge^1(\mathcal{H})$ is, for instance, the one defined by

$$\theta(\psi_1)[\dot{\psi}_2] = -\mathrm{Imag}\,\langle \psi_1, \psi_2 \rangle.$$

This shows that the geometric framework for usual Schrödinger picture is that of symplectic mechanics, as in the classical case.

A continuous vector field in $\mathcal{H}_\mathbb{R}$ is a continuous map $X\colon \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$. For instance for each $\phi \in \mathcal{H}$, the constant vector field X_ϕ defined by

$$X_\phi(\psi) = \dot{\phi}.$$

It is the generator of the one-parameter subgroup of transformations of $\mathcal{H}_\mathbb{R}$ given by

$$\Phi(t, \psi) = \psi + t\,\phi\,.$$

As another particular example of vector field consider the vector field X_A defined by the $\mathbb{C}$-linear map $A : \mathcal{H} \to \mathcal{H}$, and in particular when A is skew-selfadjoint.

With the natural identification natural of $T\mathcal{H}_\mathbb{R} \approx \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}$, X_A is given by

$$X_A : \phi \mapsto (\phi, A\phi) \in \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R} \,.$$

When $A = I$ the vector field X_I is the Liouville generator of dilations along the fibres, $\Delta = X_I$, usually denoted Δ given by $\Delta(\phi) = (\phi, \phi)$.

Given a selfadjoint operator A in $\mathcal{H}$ we can define a real function in $\mathcal{H}_\mathbb{R}$ by

$$a(\phi) = \langle \phi, A\phi \rangle \,,$$

i.e.,

$$a = \langle \Delta, X_A \rangle \,.$$

Then,

$$\begin{aligned} da_\phi(\psi) &= \frac{d}{dt} a(\phi + t\psi)_{t=0} = \frac{d}{dt} \left[\langle \phi + t\psi, A(\phi + t\psi)\rangle\right]_{|t=0} \\ &= 2\,\mathrm{Re}\,\langle \psi, A\phi\rangle = 2\,\mathrm{Imag}\,\langle -\mathrm{i}\,A\phi, \psi\rangle = \omega(-\mathrm{i}\,A\phi, \psi). \end{aligned}$$

If we recall that the Hamiltonian vector field defined by the function a is such that for each $\psi \in T_\phi\mathcal{H} = \mathcal{H}$,

$$da_\phi(\psi) = \omega(X_a(\phi), \psi) \,,$$

we see that

$$X_a(\phi) = -\mathrm{i}\, A\phi\,.$$

Therefore if A is the Hamiltonian H of a quantum system, the Schrödinger equation describing time-evolution plays the rôle of 'Hamilton equations' for the Hamiltonian dynamical system $(\mathcal{H}, \omega, h)$, where $h(\phi) = \langle \phi, H\phi \rangle$: the integral curves of X_h satisfy

$$\dot{\phi} = X_h(\phi) = -\mathrm{i}\, H\phi\,.$$

The real functions $a(\phi) = \langle \phi, A\phi \rangle$ and $b(\phi) = \langle \phi, B\phi \rangle$ corresponding to two selfadjoint operators A and B satisfy

$$\{a, b\}(\phi) = -\mathrm{i}\, \langle \phi, [A, B]\phi \rangle\,,$$

because

$$\{a, b\}(\phi) = [\omega(X_a, X_b)](\phi) = \omega_\phi(X_a(\phi), X_b(\phi)) = 2\,\mathrm{Imag}\,\langle A\phi, B\phi \rangle\,,$$

and taking into account that

$$2\,\mathrm{Imag}\,\langle A\phi, B\phi \rangle = -\mathrm{i}\,[\langle A\phi, B\phi \rangle - \langle B\phi, A\phi \rangle] = -\mathrm{i}\,[\langle \phi, AB\phi \rangle - \langle \phi, BA\phi \rangle]\,,$$

we find the above result.

In particular, on the integral curves of the vector field X_h defined by a Hamiltonian H,

$$\dot{a}(\phi) = \{a, h\}(\phi) = -\mathrm{i}\,\langle \phi, [A, H]\phi \rangle\,,$$

what is usually known as Ehrenfest theorem:

$$\frac{d}{dt}\langle \phi, A\phi \rangle = -\mathrm{i}\,\langle \phi, [A, H]\phi \rangle\,.$$

There is another relevant symmetric $(0, 2)$ tensor field which is given by the Real part of the inner product. It endows $\mathcal{H}_{\mathbb{R}}$ with a Riemann structure and we have also a complex structure J such that

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \qquad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with

$$g(Jv_1, Jv_2) = g(v_1, v_2), \qquad \omega(Jv_1, Jv_2) = \omega(v_1, v_2)\,.$$

The triplet (g, J, ω) defines a Kähler structure in $\mathcal{H}_{\mathbb{R}}$ and the symmetry group of the theory must be the unitary group $U(\mathcal{H})$ whose elements preserve the inner product, or in an alternative but equivalent way (in the finite-dimensional case), by the intersection of the orthogonal group $O(2n, \mathbb{R})$ and the symplectic group $Sp(2n, \mathbb{R})$.

The time evolution from time t_0 to time t, even in the non-autonomous case, is described in terms of the evolution operator $U(t, t_0)$:

$$\psi(t) = U(t, t_0)\psi(t_0).$$

It must be a symmetry of the theory, i.e. for each fixed t_0, $U(t, t_0)$ is a curve in the unitary group $U(\mathcal{H})$.

Assume by simplicity that $\mathcal{H}$ is finite-dimensional, and then as

$$\frac{dU(t, t_0)}{dt} \in T_{U(t,t_0)}U(\mathcal{H}) \Longrightarrow \frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} \in T_I U(\mathcal{H}) \approx \mathfrak{u}(\mathcal{H}),$$

and therefore, there exists a curve $H(t)$ in $\mathrm{Herm}(n, \mathbb{C})$ such that

$$\frac{dU(t, t_0)}{dt} = -\mathrm{i}\, H(t)\, U(t, t_0).$$

In this equation $H(t)$ does not depend on t_0 because of the relation

$$U(t, t_0) = U(t, t_1)U(t_1, t_0),$$

which implies

$$\frac{dU(t, t_0)}{dt}(U(t, t_0))^{-1} = \frac{dU(t, t_1)}{dt}(U(t, t_1))^{-1}.$$

This is a Lie system in the unitary group $U(\mathcal{H})$ with associated Lie algebra $\mathfrak{u}(\mathcal{H})$ in the most general case. Sometimes however we can deal with some of its subalgebras.

Every curve $H(t)$ in $\mathfrak{u}(\mathcal{H})$ can be written as a linear combination of at most n^2 elements, those of a basis of $\mathfrak{u}(\mathcal{H})$, and therefore these (finite-dimensional) quantum systems are Lie systems.

As the elements of the Vessiot-Guldberg Lie algebra are skew-Hermitians, all of them define simultaneously Hamiltonian vector fields and Killing vector fields, and the system is a Lie-Kähler system.

As an example consider a Hamiltonian operator $H(t)$ that can be written as a linear combination, with some t-dependent real coefficients $b_1(t), \ldots, b_r(t)$, of some Hermitian operators,

$$H(t) = \sum_{k=1}^{r} b_k(t) H_k \,,$$

where the H_k form a basis of a real finite-dimensional Lie algebra V relative to the Lie bracket of observables, i.e. $[H_j, H_k] = \sum_{l=1}^{r} \mathrm{i}\, c_{jkl} H_l$, with $c_{jkl} \in \mathbb{R}$ and $j, k, l = 1, \ldots, r$.

It determines a t-dependent Schrödinger equation

$$\frac{d\psi}{dt} = -\mathrm{i}H(t)\psi = -\mathrm{i}\sum_{k=1}^{r} b_k(t)H_k\psi.$$

The vector fields X_k such that $X_k(\psi) = -\mathrm{i}\, H_k\, \psi$ are such that the t-dependent vector vector field X corresponding to the equation is $X = \sum_{k=1}^{r} b_k(t)X_k$ and

$$[X_j, X_k] = -\sum_{l=1}^{r} c_{jkl}X_l, \qquad j,k = 1,\ldots,r.$$

As an instance, if $\mathcal{H} = \mathbb{C}^2$, the time evolution is described by a curve $-\mathrm{i}H(t) := \dot{U}_t U_t^{-1}$ in the Lie algebra $\mathfrak{u}(2)$ of $U(2)$. Using the basis

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and denoting $\mathbf{S} = (\sigma_1, \sigma_2, \sigma_3)/2$ and $\mathbf{B} := (B_1, B_2, B_3)$, the Hamiltonian can be written as

$$H(t) := B_0(t)I_0 + \mathbf{B}(t)\cdot\mathbf{S}.$$

Using the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, the Schrödinger equation is

$$\begin{pmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2B_0(t)+B_3(t) & -B_2(t) & B_1(t) \\ -2B_0(t)-B_3(t) & 0 & -B_1(t) & -B_2(t) \\ B_2(t) & B_1(t) & 0 & 2B_0(t)-B_3(t) \\ -B_1(t) & B_2(t) & B_3(t)-2B_0(t) & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

while the vector fields are now

$$\begin{aligned} X_0 &= -\Gamma = p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}, \\ X_1 &= \frac{1}{2}\left(p_2\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_2}\right), \\ X_2 &= \frac{1}{2}\left(-q_2\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial p_1} + q_1\frac{\partial}{\partial q_2} + p_1\frac{\partial}{\partial p_2}\right), \\ X_3 &= \frac{1}{2}\left(p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial p_2}\right), \end{aligned}$$

satisfying

$$[X_0, \cdot] = 0, \qquad [X_1, X_2] = -X_3, \qquad [X_2, X_3] = -X_1, \qquad [X_3, X_1] = -X_2.$$

The vector fields X_0, X_1, X_2, X_3 are Hamiltonian with Hamiltonian functions given by

$$\begin{aligned}
h_0(\psi) &= \tfrac{1}{2}\langle\psi,\psi\rangle = \tfrac{1}{2}(q_1^2+p_1^2+q_2^2+p_2^2),\\
h_1(\psi) &= \tfrac{1}{2}\langle\psi,S_1\psi\rangle = \tfrac{1}{2}(q_1q_2+p_1p_2),\\
h_2(\psi) &= \tfrac{1}{2}\langle\psi,S_2\psi\rangle = \tfrac{1}{2}(q_1p_2-p_1q_2),\\
h_3(\psi) &= \tfrac{1}{2}\langle\psi,S_3\psi\rangle = \tfrac{1}{4}(q_1^2+p_1^2-q_2^2-p_2^2).
\end{aligned}$$

h_1, h_2, h_3 are functionally independent, but $h_0^2 = 4(h_1^2+h_2^2+h_3^2)$.

When $\mathcal{H}$ is not finite-dimensional Lie system theory applies when the t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of Hamiltonians H_i closing on, under the commutator bracket, a real finite-dimensional Lie algebra.

Note however that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension.

On the other hand, as the fundamental concept for measurements is the expectation value of observables, two vectors ψ_1 and ψ_2 of $\mathcal{H}$ such that

$$\frac{\langle\psi_2, A\psi_2\rangle}{\langle\psi_2,\psi_2\rangle} = \frac{\langle\psi_1, A\psi_1\rangle}{\langle\psi_1,\psi_1\rangle}, \quad \forall A \in \mathrm{Her}(\mathcal{H}),$$

should be considered as indistinguishable.

This is only possible when ψ_2 is proportional to ψ_1, and therefore we must consider rays rather than vectors the elements describing the quantum states.

The space of states is not $\mathbb{C}^n$ but the projective space $\mathbb{CP}^{n-1}$.

It is possible to define a Kähler structure on $\mathbb{CP}^{n-1}$ and therefore to study Lie-Kähler systems leading to superposition rules and to study time evolution in this projective space.

Lie Systems and Schrödinger equation

A linear SODE in normal form $\phi'' = b_1(x)\phi + b_2(x)\phi'$ can be written in the form of a system of two first-order differential equations in the variables (v_ϕ, ϕ):

$$\left\{ \begin{array}{rcl} v'_\phi & = & b_2(x)v_\phi + b_1(x)\phi \\ \phi' & = & v_\phi \end{array} \right. .$$

Identifying $\mathbb{R}^2$ with $T\mathbb{R}$, (v_ϕ, ϕ) are bundle coordinates, the preceding system determines the integral curves of the x-dependent vector field

$$X = v_\phi \frac{\partial}{\partial \phi} + (b_1(x)\phi + b_2(x)v_\phi) \frac{\partial}{\partial v_\phi},$$

which is said to be a SODE vector field because of the coefficient of $\partial/\partial\phi$.

The linear system determining its integral curves is

$$\begin{pmatrix} v'_\phi \\ \phi' \end{pmatrix} = \begin{pmatrix} b_2(x) & b_1(x) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_\phi \\ \phi \end{pmatrix} .$$

The projection onto $\mathbb{R}$ of such curves are solutions of the differential equation

$$\phi'' = b_2(x)\phi' + b_1(x)\phi.$$

We are mainly interested in equations of Schrödinger type, those with $b_2(x) \equiv 0$.

The corresponding vector field is a linear combination $X = b_1(x)X_1 - X_3$ where

$$X_1 = \phi\,\frac{\partial}{\partial v_\phi}\,, \qquad X_3 = -v_\phi\,\frac{\partial}{\partial \phi},$$

which together with

$$X_2 = \frac{1}{2}\left(v_\phi\frac{\partial}{\partial v_\phi} - \phi\frac{\partial}{\partial \phi}\right),$$

close on a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$:

$$[X_1, X_3] = 2\,X_2, \qquad [X_1, X_2] = X_1, \qquad [X_3, X_2] = -X_3\,.$$

Therefore Schrödinger type equations and the corresponding linear systems are Lie systems with Vessiot-Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Vector fields X_1, X_2 and X_3 are fundamental vector fields corresponding to the linear action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$.

The map $F : \mathbb{R}^2_* \to \mathbb{R}$ defined by $F(x,y) = x/y$ is equivariant with respect to the the restriction of the linear action Φ of $SL(2,\mathbb{R})$ on $\mathbb{R}^2_*$ and the action Ψ of $SL(2,\mathbb{R})$

on $\mathbb{R}$, or even better on the real projective line $\mathbb{R}P^1 = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, by linear fractional transformations, i.e. $\Psi : SL(2, \mathbb{R}) \times \mathbb{R}P^1 \to \mathbb{R}P^1$ is defined by

$$\begin{aligned} \Psi(A\,, u) &= \frac{\alpha\, u + \beta}{\gamma\, u + \delta}\,, \qquad \text{if} \;\; u \neq -\frac{\delta}{\gamma}\,, \\ \Psi(A\,, \infty) &= \frac{\alpha}{\gamma}\,, \qquad \Psi\left(A\,, -\frac{\delta}{\gamma}\right) = \infty\,, \\ A &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R})\,. \end{aligned}$$

Equivariance means that $F \circ \Phi_A = \Psi_A \circ F$. The corresponding fundamental vector fields of the action Ψ are now

$$\overline{X}_1 = \frac{\partial}{\partial u}\,, \quad \overline{X}_2 = u\frac{\partial}{\partial u}\,, \quad \overline{X}_3 = u^2\frac{\partial}{\partial u}\,,$$

and as F is equivariant, the fundamental vector fields associated to Φ and Ψ are F-related, i.e. $\overline{X}_i = F_*(X_i)$, $i = 1, 2, 3$, and then a system defined by the vector fields $\overline{X}_i$ is a Lie system corresponding to a Riccati equation.

The image under F of an integral curve of the x-dependent vector field $X = b_1(x)\, X_1 + b_2(x)\, X_2 + b_3(x)\, X_3$, which is a linear system, is an integral curve of $\bar{X} = \bar{b}_1(x)\, X_1 + \bar{b}_2(x)\, X_2 + \bar{b}_3(x)\, X_3$, i.e. a solution of the corresponding Riccati equation.

We can also consider a new vector field

$$X_4 = \frac{1}{2}\left(v_\phi \frac{\partial}{\partial v_\phi} + \phi \frac{\partial}{\partial \phi}\right),$$

which commutes with X_1, X_2 and X_3 and they generate the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$.

The integral curves of a generic Lie system with such a Vessiot-Lie algebra, namely $X = b_1(x)\,X_1 + b_2(x)\,X_2 + b_3(x)\,X_3 + b_4(x)\,X_4$, are determined by the system

$$\begin{pmatrix} v'_\phi \\ \phi' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(b_2(x) + b_4(x)) & b_1(x) \\ -b_3(x) & \frac{1}{2}(b_2(x) - b_4(x)) \end{pmatrix} \begin{pmatrix} v_\phi \\ \phi \end{pmatrix}.$$

Note also that $\overline{X}_4 = F_*(X_4) = 0$. A SODE vector field is a particular case corresponding to the choice $b_3 = -1$ and $b_2 = b_4$.

We can use that the group of curves in a Lie group G acts on the set Lie systems with associated Vessiot-Lie algebra $\mathfrak{g}$ to reduce a given Lie system to another one of the same type. In our case, to relate Schrödinger type equations with different potentials by means of curves in $GL(2,\mathbb{R})$. This is the essence of Darboux transformation method.

The advantage to see Schrödinger equations as Lie systems with Vessiot-Lie algebra $\mathfrak{gl}(2,\mathbb{R})$ instead of $\mathfrak{sl}(2,\mathbb{R})$ is that we can transform by curves which are in $GL(2,\mathbb{R})$ but not in $SL(2,\mathbb{R})$.

Remark that if we are interested in taking into account the tangent bundle character of $T\mathbb{R}$, we should consider transformations induced from those of the base manifold.

So, given a strictly positive function φ_0 we can associate the function ϕ with the new function $\bar{\phi}$ by means of $\phi = \varphi_0\,\bar{\phi}$. This induces a transformation

$$\begin{pmatrix} v_\phi \\ \phi \end{pmatrix} = \begin{pmatrix} \varphi_0(x) & \varphi_0'(x) \\ 0 & \varphi_0(x) \end{pmatrix} \begin{pmatrix} v_{\bar{\phi}} \\ \bar{\phi} \end{pmatrix}.$$

If ϕ is a solution of $\phi'' = b_1(x)\phi + b_2(x)\phi'$, then $\bar{\phi}$ satisfies

$$\varphi_0(x)\bar{\phi}'' + (2\varphi_0'(x) - b_2(x)\varphi_0(x))\bar{\phi}' + (\varphi_0'' - b_1(x)\varphi_0 - b_2(x)\varphi_0')\bar{\phi} = 0,$$

and then when $\varphi_0(x)$ is a particular solution of the given equation, the transformed equation reduces to $\varphi_0(x)\,\bar{\phi}'' + (2\,\varphi_0'(x) - b_2\,\varphi_0(x))\bar{\phi}' = 0$, in which the dependent variable $\bar{\phi}$ is absent, which is quickly integrated by order reduction.

This is an explicit example of reduction procedure for Lie system when a particular solution is known.

The transformation $\phi = \varphi_0\,\bar{\phi}$ is very useful in factorization problems, the differential operator $\partial/\partial x$ becoming a ladder-like operator $\partial/\partial x - \varphi_0'/\varphi_0$. This allows to introduce factorisable Hamiltonians, their partners leading to supersymmetric quantum mechanics and interesting relations among their spectra, in particular special methods for introducing or removing eigenvalues and eigenstates.

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